Tier II Real Estate

Roger J. Brown

The Pennsylvania State University
409 Business Administration Building
University Park, PA 16802
(814) 865-4172
rjb21@psu.edu
http://www2.smeal.psu.edu/courses/rest301.brown/home.html

IMOJIM Investments
PO Box 1146
Alpine, CA 91903-1146
(619) 445-3405
rjb21@home.com
http://members.home.net/rjb21

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1. Introduction:

Most of the study of private investor activity to date has concentrated on investment in financial assets. Microeconomic theory of consumer choice and decision-making under uncertainty is highly developed and finds much empirical support in stock market data. Data providing similar evidence for direct, private investment in real property have not been easy to obtain. As a result there is a paucity of research in this area. Still, it appears that many private investors eschew the stock market in favor of owning parcels of real property. Why they do this is unclear. A simple explanation would be that real property investment offers a higher risk-adjusted return than financial assets. There is little evidence that this is so.

Most research on real estate portfolios has involved investors whose capital sources make forming a portfolio a practical reality. Thus, studies abound regarding major institutional investors and the indices formed from the returns earned by very large properties.

Examining the difference between individual and institutional investing, Markowitz (1991b) confesses "the 'investing institution' which I had most in mind when developing portfolio theory for my dissertation was the open-end investment company or 'mutual fund'" (p. 2). Markowitz recites a host of problems facing the individual that simply don't apply to the institution, concluding that simulation is perhaps the only practical way to implement portfolio analysis at the individual level.

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This article investigates the return distributions of economic agents who choose to invest their discretionary funds in a parcel of income-producing real property rather than in financial instruments. Thus, it must be remembered that what is examined here is not Professor Markowitz' "investing institution". Nor is it a real estate institution such as a real estate investment trust (REIT) or pension fund. Rather, it is the individual private investor. The finance literature canonizes the maxim “Don’t put all your eggs in one basket” by applying Modern Portfolio Theory (MPT) to financial assets. The alternative, “Put all your eggs in one basket – *and watch that basket*”, a theory of private real estate investment, is the subject of this paper.

To distinguish this research, real estate is divided into three categories, or Tiers. Tier I is made up of the small property apartment market, more specifically parcels of real property containing from one to four dwelling units, one of which is very often owner occupied. Tier I is thus separate and distinct from the market for investment real estate. The investment real estate market is further divided by size into non-institutional and institutional components, constituting Tiers II and III, respectively. This paper is exclusively interested in Tier II and by its data set is further restricted to multi-family residential property in Tier II.

For institutional reasons, most having to do with financing, it is convenient to assume a boundary of 4 dwelling units on a single parcel as the upper limit of Tier I. Brown (2000) provides evidence suggesting that the lower boundary of Tier III for residential property is 100 dwelling units on a single parcel. The remaining property, for the purposes of this paper apartments comprising more than four and less than 100 dwelling units, is classified Tier II.

Practically every first semester real estate textbook contains a list of reasons, sometimes called “the Laundry List”, that real estate is different from financial assets. Such a list is tacit admission that (i) the real estate market is inefficient and (ii) applying portfolio theory to real estate is, at best, a challenging task. For the individual with limited resources, practical considerations such as high down payment requirement, high transaction costs and lack of liquidity often place forming a portfolio of private real estate assets out of reach. Accordingly, a central value of the Markowitz model – being able to diversify away non-systematic risk – is unavailable to such an investor. If we suspend the practical problems and assume the investor *is* able to form portfolios, another difficulty arises: These may not be *efficient set* portfolios in the Markowitz sense. The reason for this requires a digression into the statistical theory underlying MPT.

Modern Portfolio Theory is based on a set of strong assumptions, one of the most important being that returns are normally distributed. This assumption permits a finite variance, making possible the covariance and correlation coefficients needed to implement MPT. When asset returns are not perfectly correlated, combining them into portfolios is more "efficient" than holding individual assets. When assets are combined, the quadratic nature of asset covariances creates a curve known as the efficient frontier. Figure 1 illustrates how the frontier becomes more convex to the y-axis as the correlation coefficient (derived from the covariance) declines from 1. When correlation is perfect, that is it equals 1, the far right plot is linear because the quadratic term is zero. Return is then perfectly linear in risk and there are no gains to be had from diversification. Reading the plots from right to left illustrates how gains from diversification increase as covariance declines. As the plot becomes more convex, more return may be obtained for the same amount of risk or, alternatively, given a fixed return there are opportunities to reduce risk by combining assets.

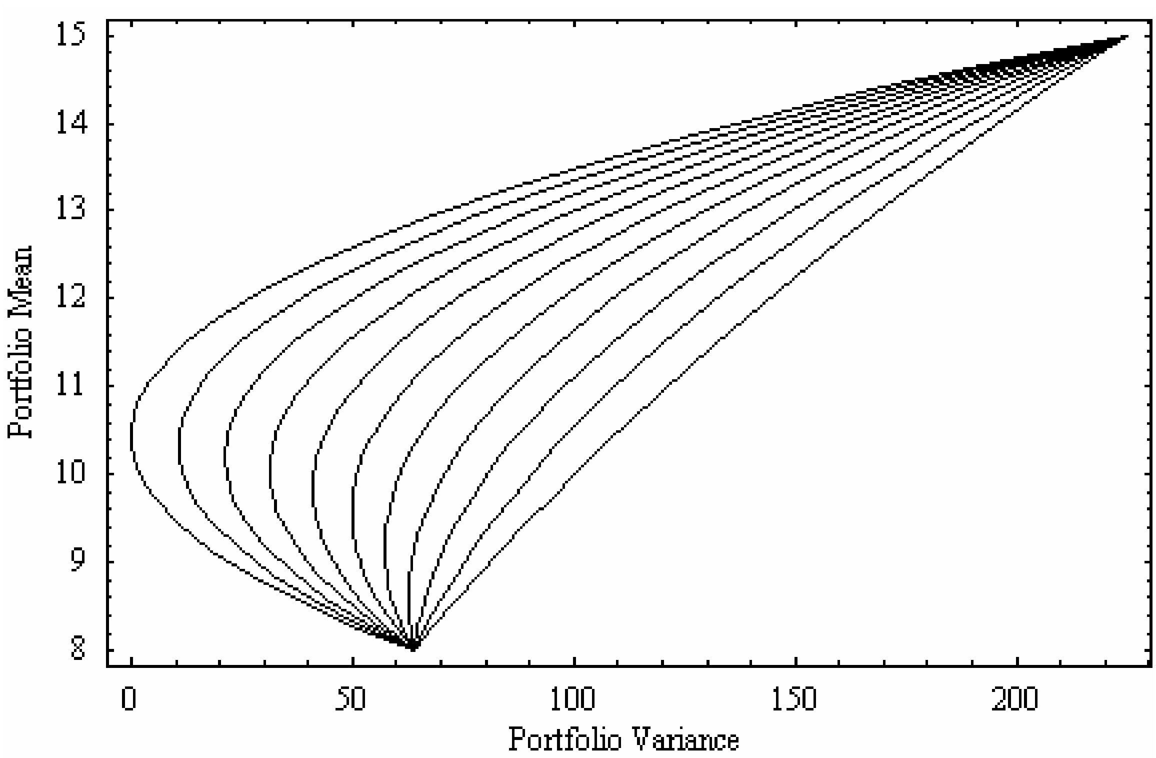

Figure 1*:* Efficient frontiers reflecting different correlation coefficients ranging, left to right, from -1 to 1 in increments of .2

A major insight flowing from MPT is what is known as "the mean-variance rule." If investors make decisions based solely on mean and variance, the practical consequence is that the first two moments of the distribution are sufficient to form efficient portfolios. Another important ramification is that gains from diversification are available, allowing investors to "diversify away" asset-specific risk.

Many authors in finance (Mandelbrot 1963a and 1963b, Fama 1963, 1965a and 1965b, Blume, 1970, Fama and Roll, 1971 and Fama and Miller, 1972, Peters, 1996) and a few authors in real estate (Myer and Webb, 1994, Young and Graff, 1995, Graff, Harrington and Young, 1997 and 1999 and Brown, 2000) have questioned the assumption of normality. The finance authors found distribution data for financial assets sufficiently close to normal to produce reasonable empirical results. The real estate authors found this not to be the case.

The normal distribution is actually a *special case* of a family of distributions known as "stable Paretian" (SP) distributions. These distributions are unimodal, retain their shape under addition and weighted addition (hence "stable") and asymptotically approach the Pareto distribution (hence "Paretian"). Of the infinite number of possible SP distributions, there are only three with known, closed form, probability density functions (pdf). The first is the normal distribution, the second is the arctangent or Cauchy distribution, and the third is the Levy distribution. Due to their range, the Cauchy and Levy distributions are of little interest in a financial setting. With those exceptions noted, unless the SP distribution is normal, in the limit no moments above the first exist for a broad range of these otherwise useful distributions. No second moment means no variance, without which there can be no covariance. Without covariance one cannot form efficient set portfolios in the Markowitz sense.

This condition is not fatal. Even if the pdf does not exist, all distributions – including the entire range of Stable distributions – have a characteristic function (ch.f.). Stable distributions are fully characterized by four parameters: $\alpha \in (0,2]$, is known as the *characteristic exponent* or *index of stability*; $\beta \in [-1,1]$, the *skewness parameter*, is symmetric around the mode when $\beta = 0$; $\gamma > 0$, is a *scale parameter* expanding and contracting the distribution around its mode; and $\delta \in (-\infty,\infty)$, is the *location parameter*, shifting the distribution left and right along the x-axis.

The most important of these four is α, as it determines the fatness of the tails. The value of α is very useful for assessing risk in non-normal situations. When $\alpha = 2$ the Stable distribution is also the normal. As the value of α declines toward 1 the tail of the distribution becomes fatter, revealing more outcomes away from the mean.[1] The second most important parameter is the skewness of the distribution as represented by the β parameter. Equally vital is the interaction of β with α, an interaction more fully discussed in Section 3 below.

To date, the conclusion is that real estate data is non-normal "fat tailed" to a degree that tests of real estate performance within the finance paradigm yield unacceptable empirical results. Until now only institutional (Tier III) real estate has been tested and its data has evidenced distributions with heavy tails on either left or right ($-1 > \beta > 1$). This paper will provide theory and empirical support for a claim that *private* real estate investment return distributions are heavy *right* tailed ($0 < \beta < 1$).

[1] We generally ignore α values below 1. According to McCulloch (1978) "[i]n financial applications it is universally assumed that $1 < \alpha < 2$" (p. 601 fn 1)

2. Toward a theory of private real estate investment

The basic unit of return measurement is the incremental difference between beginning and ending values,

$$R = \frac{P_t}{P_{t-1}} - 1 \quad , \tag{1}$$

where R is return, P_t is today's price and P_{t-1} is some price in the past. The investor in financial assets has very little to do with how today's price of a stock differs from yesterday's.[2] Private real estate offers the opportunity for an investor to "have a say" in its value by adding his own management to the equation. Thus, by adding entrepreneurial labor to his investment, the private real estate investor/operator may positively influence P_t and achieve a greater return. Of course, this means that a portion of the return we observe represents a return on the investor's time as well as on his invested capital. Such a reality adds yet another reason to the list describing why real estate is different.

From this I theorize four reasons private real estate return distributions should be fat *right* tailed. They are:

(a) Real estate involves a fixed resource. The land component of real property is fixed in supply. In an urban setting having growth controls the supply may be further constrained but it is at least true that "They ain't making any more".[3]

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[2] Unless one considers the actions of buying and selling in which case the aggregate action of many buyers influence stock price. However, a single individual deciding to buy or sell a small quantity of stock is the classic "price taker".

[3] This comment about the wisdom of buying land is attributed to a number of sages, most frequently to Will Rogers. In the "no growth" atmosphere of many of today's cities Will might well elaborate on his earlier remark to say "They ain't making any more and they are making less of it legal everyday."

(b) Investors extended their holding periods. Because of items on the Laundry List such as high transaction costs and site specific financing, real estate investors are resigned to long holding periods. If the real estate cycle is a long one the thoughtful investor has not only an opportunity to choose the most beneficial time to liquidate (as does any investor) but he is restrained by institutional factors from the hasty, emotional or ill-advised liquidation.[4]

(c) There is an α/β tradeoff at work. There is an offsetting interplay between the α and β parameters of the Stable distribution. Below it is shown that investor utility remains constant as α falls if at the same time β is rising.

(d) Investors add labor. Private real estate investment returns contain a component representing the owner's labor in the form of management contribution.

The first of these, item (a), is not only the subject of folklore but is supported by the most basic Marshallian supply and demand analysis. Item (b) has both intuitive appeal and some empirical support (Brown 2000). These are not the focus of this paper. Insights relating to items (c) and (d) are developed in detail in Sections 3 and 4 below.

3. The α/β tradeoff.

Perhaps one of the most complex notions of the interconnectedness of modern society is the relationship between risk and reward. It is remarkable that Markowitz (1952) and Sharpe (1963) were able to simplify it to a linear relationship and that their simplification has survived for so long. The literature of the past 40 years reveals the many blows it has taken in the course of justly earning the title "Dominant Paradigm" in finance.

It is not clear that MPT should be the dominant paradigm of real estate. It may be that defining risk as the second moment of a two-moment distribution is too simplistic for real

[4] This assumes the investor is unencumbered and controls his own destiny. Certainly the pressure of leverage can force sales at inopportune times but that reality goes beyond the scope of this paper.

estate.[5] The notion that the investor's trade off is limited to statistical mean and variance may not tell the complete picture of real estate. A further investigation of the more general idea of variance is in order.[6] Indeed, an enlarged view of the concept of "variance" may be necessary to capture what risk means to the private real estate investor. To begin, consider the symmetrical shape of the normal distribution in Figure 2

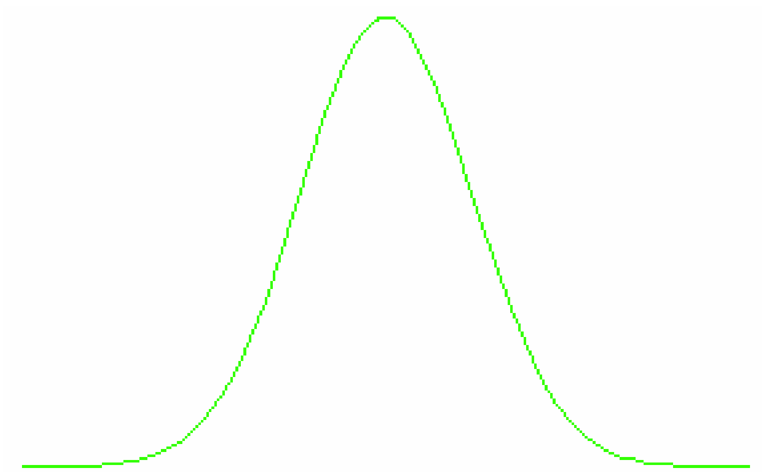

Figure 2: The normal probability distribution

Comparing to the shape of the symmetrical fat tailed Stable distribution in Figure 3, at first glance one sees little difference until the two are super-imposed over each other as shown in

[5] In fact, it may be too simplistic for *any* asset in the 21st Century. In 1952 no one imagined that virtually every thinking investor would someday have 64 MB of RAM sitting on his desk with the capability to measure performance by a host of different numerical and graphical ways that go far beyond the analytical techniques available in the 1950s.

[6] The language becomes unavoidably imprecise here. Most of the ensuing reference to "variance" is intended to include such broader definitions of "more possible outcomes" rather than the strict statistical/mathematical meaning of variance as deviation from the mean. Such a broader meaning is intended to capture those outcomes visible in a stable distribution with fat tails that are not shown by a normal distribution.

Figure 4, which displays the additional outcomes visible in the SP distribution that are ignored by the normal.

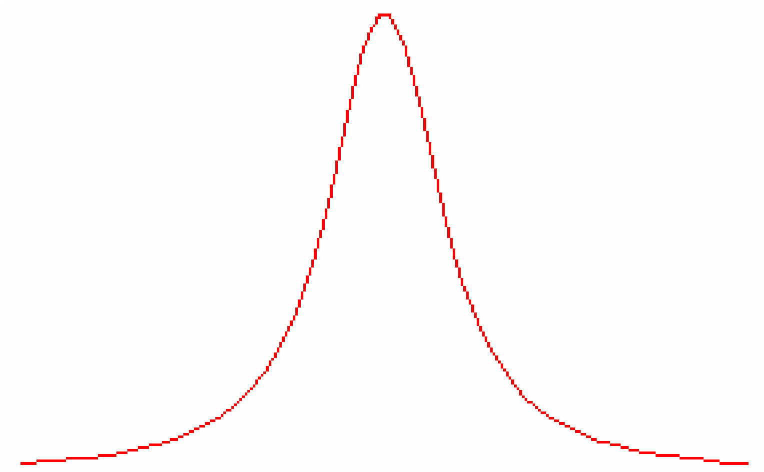

Figure 3: A "heavy-tailed" distribution

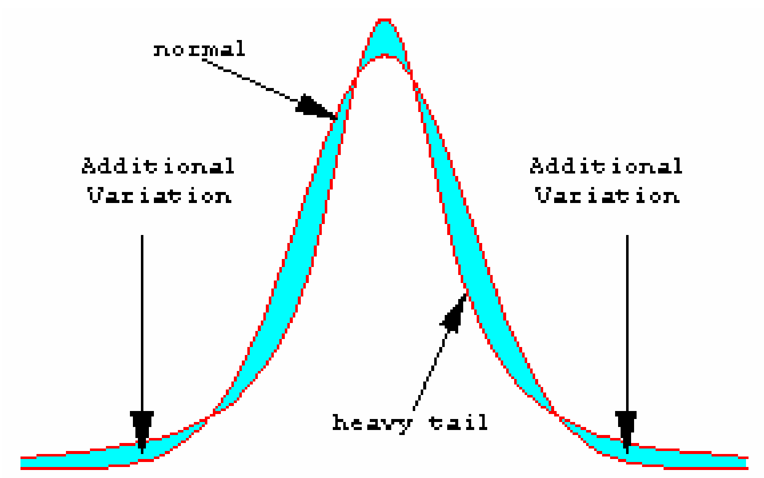

Figure 4: The normal and the heavy-tailed distribution on a single plot

As McCulloch (1996) observed "If asset returns are truly governed by …stable distributions, life is fundamentally riskier than in a Gaussian world." (p. 393). Given that real estate return distributions exhibit fat tails, hence more possible outcomes that may be seen as additional risk, we return to the question: Why should one invest in these assets if assets with normally distributed returns exist? If real estate investors are rational risk-averse utility maximizers like other agents there must be some other trade off at work in this market to compensate these investors for the additional risk they assume. I hypothesize that one form of compensation would be positive skewness. For the Stable distribution this means its β parameter is positive. Clearly, variation on the *positive* side of the distribution is more tolerable than the alternative. Hence, the effect of a positive β is to take probability mass from the left side and deposit it on the right side. Figure 5 adds a third distribution to the plot in Figure 4, this one Stable and positively skewed (having a positive β), reflecting a fat *right* tail.

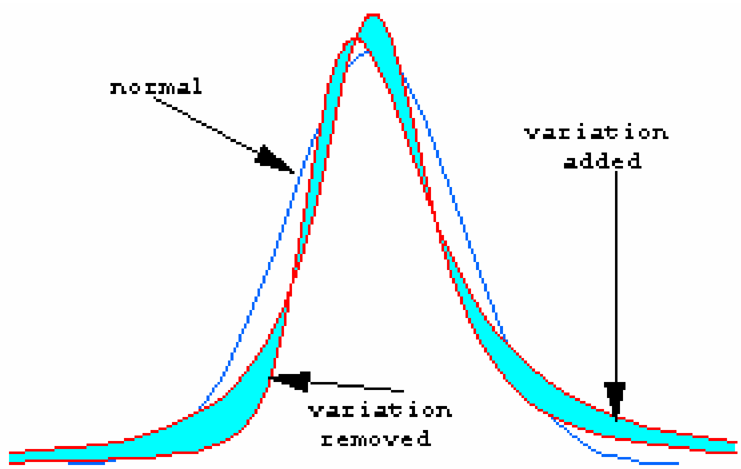

Figure 5: Three distributions, the normal, the symmetric stable and the right skewed stable

The assumption of Normality not only permits many of the results of MPT, it also permits the form of those results to be mathematically tractable in the classic sense. That is, the existence of a pdf allows a closed form analytical solution. The absence of a pdf requires numerical and graphical demonstrations in lieu of analytical solutions. With the advent of fast, cheap computing power, numerical and graphical methods of problem solving lately have enjoyed increased favor. Recently, Judd (1998) and Levy, Levy and Solomon (2000) have made important contributions to, respectively, economics and finance.[7] Brown (2000) describes the means of “simulating” a pdf for the Stable distribution using the fact that the Stable ch.f. and its simulated pdf constitute a Fourier Transform pair.[8] This method is employed to produce the graphical representations here.

Following the procedures of the classics works on decision-making under uncertainty (Pratt (1964) and Arrow (1965)), it is assumed that investors are rational utility maximizers who view risky alternatives as choosing between lotteries. Each possible return on investment is a “draw” from one of the various available lotteries. Different lotteries are described by the shape of their distribution. I assert that the four parameters of the SP distribution do a better job of describing the lottery known as private real estate investment. It is further assumed that investors produce utility by the negative exponential function

$$u(x) = 1 - \left(e^{-.1x}\right) \qquad (2)$$

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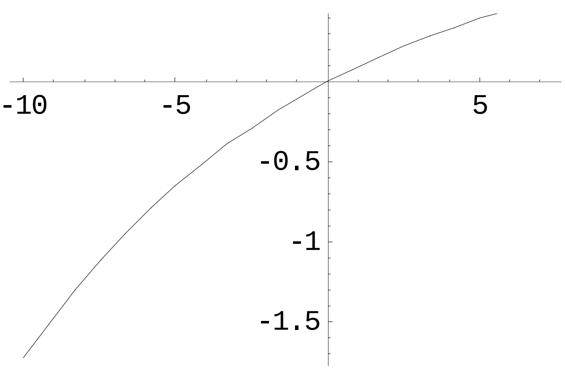
[7] Commenting on Levy, Levy and Solomon (2000), Markowitz observed "If we restrict ourselves to models which can be solved analytically, we will be modeling for our mutual entertainment, not to maximize explanatory or predictive power."

[8] Granger and Orr (1972) note that "Much of the work that ordinarily uses probability or frequency functions can be carried out in the transform space of the characteristic function" (p.275).

where x represents a rate of return presumed to be drawn from a lottery of returns and e is the base of the natural logarithm. This function has two desirable properties, as illustrated in *Figure 6*. One, the function is zero when x is zero, an intuitively appealing quality in that investors are probably ambivalent about breaking even.[9] Two, its second derivative is negative, conforming to the conventional belief in diminishing marginal returns.[10]

$$\partial_{xx} = -.01\left(e^{-0.1x}\right) < 0 \quad . \tag{3}$$



$$u(x) = 1 - \left(e^{-.1x}\right)$$

Figure 6: The negative exponential utility function.

[9] This is a mere convenience. No generality is lost by the shift of the function in this manner.
[10] Tsaing (1972) presents two additional properties that are also desirable. These have to do with whether the utility function exhibits constant absolute or relative risk aversion.

As paper representation constrains the number of dimensions displayed by any illustration, it is necessary to fix two of the four parameters of the SP ch.f. in order to plot. Here γ is fixed at 1 and δ at .06, permitting a plot of changes in utility with changes in α and β under these bounded conditions. *Figure 7* and *Figure 8*, show that utility rises as α and β each rise.

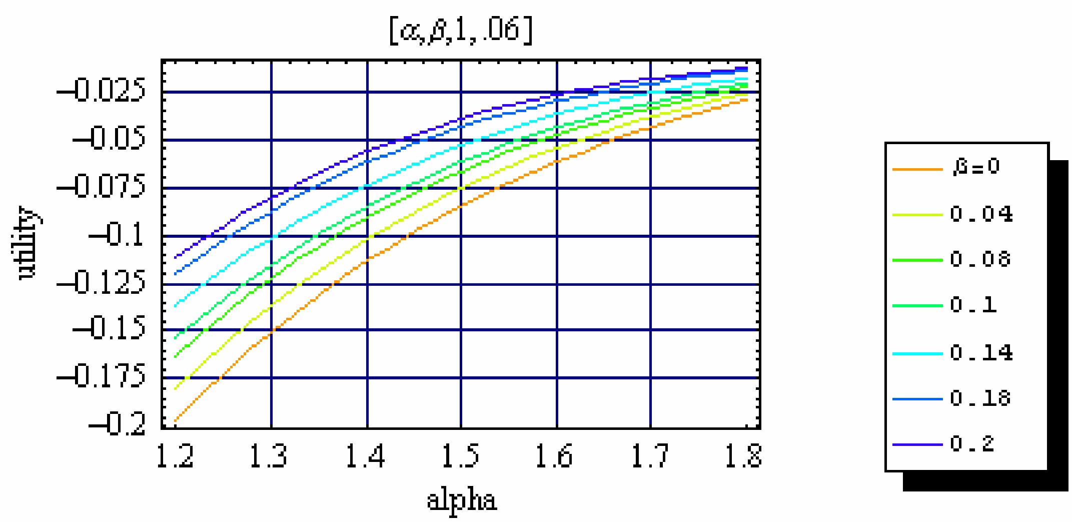


Figure 7: Utility rises as α rises toward 2 (the normal distribution)

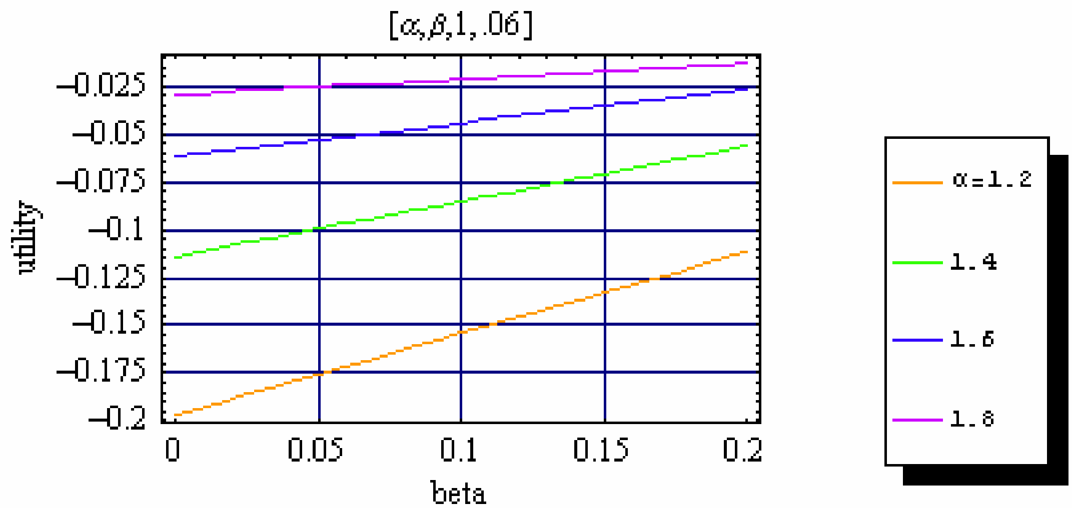


Figure 8: Utility rises as beta move away from zero in a positive direction

Recall that the tail of the SP distribution becomes *thinner as α rises*. It is at its thinnest when α = 2 where the distribution is normal. Interest here is in change of α in the opposite direction - the increased risk associated with *a fall* in α. Hence, one must reverse one's perspective and reconsider *Figure 7*, looking at effect on utility as α falls (reading the plot from right to left) and the tail becomes fatter.

The foregoing offered as a graphic evidence, there is another way to view this problem. We may compute the values of α and β under conditions of fixed utility and observe how they change in relation to each other. *Table 1* shows values of α and β when utility is fixed at -.05 (γ

= 1 and δ = .06 as before). Note the inverse relationship between α and β given a fixed utility. The theory predicts that as alpha falls, beta must rise to keep utility constant. This is the tabular form of the effects shown in Figures 7 and 8. The intuition that investors will accept fat tails if they improve their chance of a positive draw.

Table 1: The changing values of α and β with utility held constant at -.05

α	β	γ	δ
1.66204	0	1	.06
1.64485	.02	1	.06
1.62649	.04	1	.06
1.60691	.06	1	.06
1.58606	.08	1	.06
1.56393	.10	1	.06
1.54051	.12	1	.06
1.51581	.14	1	.06
1.48986	.16	1	.06
1.46270	.18	1	.06
1.43441	.20	1	.06

A final graphical look at this analysis is the contour plot in *Figure 9* where utility is shown constant as α and β change. Logically, utility is highest in the northeast quadrant when α approaches 2 and the skew is at the high end of the range. By following an almost vertical strip on the plot, one can see how falling α is offset by rising β, holding utility constant.

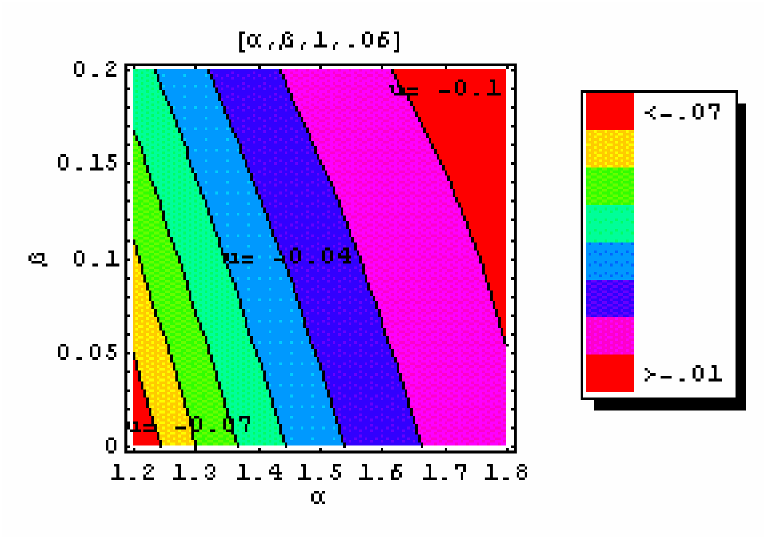


Figure 9: Contour plot of constant utility as α and β change

4. The labor component

<u>Foundation for the model</u>

The simple static model of labor supply (Killingsworth, 1983) holds that one chooses between work and recreation. To the extent one works for an employer and receives wages, the static model assumes that one earns money, permitting one to purchase goods and services, all of which are consumed in the same period. If one chooses recreation rather than work, enjoying leisure, the implicit price of leisure is the forgone consumption goods one may have gained by working. The theory claims that most people enjoy a mix of work (consumption goods) and recreation.

Acknowledged in the simple static model is that fact that an agent may have or acquire along the way property in the form of bank accounts that pay interest or stock that pays dividends. The agent's reservation wage is affected by the amount of property income. The natural conclusion is that "coupon clippers" cannot be hired because their marginal substitution of wages for leisure is too high due to the fact that their non-human assets provide their creature comforts without the necessity of a job or an employer.

Adding real estate introduces daunting complexity. It is common for labor theorists to use financial assets as property, but when the source of property income is *real* property, interesting things can and do happen. The important distinction is that the investor may impact the productivity of his investment if it is a real asset. Effort applied to the operation and management of the asset can change the income from the property and, therefore, the outcome of the investment.

When the investor adds his labor to the property he owns there are several new considerations. First, the time for that labor must come from somewhere. Previously all time was devoted to work and leisure. With a fixed amount of time available, real estate investment activity much take from one or the other. Second, the election to work on his property to improve its income introduces a third choice variable in the utility function. Third, the addition of the owner's labor normally represents a positive value, increasing property income *over time*.[11] Hence, there is a time dimension leading to the modeling difficulties associated with a dynamic setting. Fourth, any model must provide for an efficiency factor to allow for differing skills among real property owners. Presumably, investors self-select into this market based on some

[11] Of course the opposite is possible, the addition of some owners' effort no doubt lower property income.

real or imagined ability to successfully operate their acquisition. Those with particularly good skills should enjoy improved results as a consequence of greater talent. Those misjudging their abilities or simply lacking what it takes should have marginally lower returns and/or be washed out of the market in sales to more efficient agents.

A host of other, smaller, considerations must be incorporated. The balance required is to make the model simple enough to be as universal and elegant as possible, yet include the vital, often complicating, factors necessary to make it realistic. Some of these are housekeeping items. For instance, it is reasonable to assume that the investor enters the real estate investment market only after amassing enough wealth to overcome the down payment barrier. It is also assumed that the investor has a retirement goal. In fact, the argument here is that one of the reasons the investor chooses real estate is to accelerate the date of retirement to an earlier point in life. This introduces more differences from the traditional labor model. In labor supply theory it is assumed that the length of economic life is measured by the time between the agent first enters the market, perhaps after graduating from school, until death. The model below assumes that (1) economic life begins at the time real estate is purchased, presumably after some period of wealth accumulation time following entry into the labor force leading up to that point when capital for a down payment is available;[12] and (2) that some retirement date prior to death, a critical factor in the agent's plan, exists. This retirement date is variable and based, in part, on the investor's efficient operation of his real estate. This, of course, means that the time spent

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[12] This period can be very short if one believes the kind of "Nothing Down" stories that appear on late night television infomercials.

working for an employer and the time (retirement) following that working period are both variable but still sum to the whole of economic life as defined here.

The matter of bequests must be addressed. One method, the one employed here and quaintly referred to as “the die broke scheme”, assumes bequests = 0. That is, all wealth is consumed during one’s lifetime. The alternate extreme, for which competing monikers might be “the live forever scheme” or more likely “the leave a rich wife scheme”, forbids any capital erosion. Essentially a constant wealth alternative, in such a model only earnings are consumed after retirement while capital remains intact. Neither of these extremes is terribly realistic and something in between is what we observe in practice but the model demands some simplifying assumptions so the zero bequest assumption will be maintained here.[13]

Special note must be made of the efficiency factor. This may be viewed as the productivity of the investor’s entrepreneurial effort applied to his rental property (how good he is at it). This appears in the model below as a function that acts like (but is not) a wage rate. Unlike wages sometimes behave, it is neither constant nor monotonically increasing over time. There are economies of scale, discontinuities and variation, all differing over time, which affect how much the investor makes on any particular property over any specific holding period. A useful characterization, but still an oversimplification, would be to call it the average rate at which the residual claim is rewarded over the holding period. This reward is only in part for time spent. It is also a reward for bearing the *uncertainty* associated with the residual claim.[14]

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[13] The electronic version of this paper, available to be downloaded from the author’s web site contains different demonstrations of alternative sets of assumptions in the form of Mathematica notebooks.

[14] The seminal work on the nature of risk and uncertainty is Knight (1921). Risk, according to Knight, is a subset of uncertainty that involves a known, *a priori* probability distribution. The risk bearing may be treated as a cost that

It is important to note that the efficiency factor does _not_ represent compensation for property management but for the larger matters of strategy, timing and entity management. This is not to say that the investor might not also choose to do his own property management. He may also choose to do his own plumbing. Presumably these decisions will be determined by whether he has comparative advantages in these areas. No assumption is made that he MUST hire vendors for these tasks or that he is necessarily better able to perform them. The primary purpose of the efficiency factor is to represent the payoff to the residual claim arising from the global coordination of the entire venture, of which property management, maintenance, etc. are all a part but do not, in the aggregate, represent the whole of what is required to own and successfully operate real property.

The model

The following notations will be used in the model:

W_0	=	Initial wealth at t_{Wo}, assumed > 0 because of the down payment barrier
$w_1(t)$	=	Wealth accumulation function for working (pre-retirement) years
$w_2(t)$	=	Wealth accumulation function for retirement years
w0	=	Boundary condition in w_2, the differential equation for wealth production after retirement. w0 is the wealth accumulated at the end of w_1.
wg	=	Wage rate applied to H during time employed
t	=	time
T	=	Total time working and retired (t_w+t_r). All of economic life
H	=	Exogenously determined portion of one's time devoted to working at a job
L	=	Portion of time devoted to leisure, assumed > 0 because at least some time must be spent sleeping

may be disposed of by the payment of an insurance premium out of operating expenses. The reward for bearing uncertainty is the residual or profit the owner receives after all other claims are satisfied.

S	=	Portion of time devoted to working on real property investments, the time remaining after H and L. Note that H + L + S = 1
t_{Wo}	=	Beginning point of working years. Also beginning point of economic life and real estate investment acquisition point
t_w	=	Working time or pre-retirement time in years
t_r	=	Retirement time in years, ending at death (assumes bequests = 0 or W = 0 at death)
c_w	=	Consumption during working years
c_r	=	Consumption during retirement years
r	=	Interest rate or rate of return over time
p	=	Exogenously given prices
ρ	=	Efficiency of time spent on real property investment. Productivity of entrepreneurial effort
α	=	Preference for consumption over a base amount, assumed to be $0 < \alpha < 1$
β	=	Preference for leisure over a base amount, assumed to be $0 < \beta < 1$
γ	=	Preference for time retired, assumed to be $0 < \gamma < 1$
uf	=	Utility function of the form $u(xyz) = x^{\alpha}y^{\beta}z^{\gamma}$

Wealth is accumulated during working years by the function w_1, derived from the differential equation

$$\frac{d}{dt}w_1(t) = w_1(t)\rho\frac{S}{S+1} + wgH - c_w p \tag{4}$$

where the boundary condition, initial wealth = W_0, is increased by efficient time spent operating real estate, by wages times the proportion of one's spent time working for an employer and decreased by consumption times prices. The real estate portion of time appears as a function, $\left(\frac{S}{S+1}\right)$, affected by efficiency (ρ) and evidencing diminishing marginal returns. One of the solutions to (4) is

$$w_1(t) = \frac{c_w p + c_w pS - Hwg - HSwg + e^{\frac{St_w\rho}{1+S}}\left(-c_w p(1+S) + H(1+S)wg + SW_0\rho\right)}{S\rho} \tag{5}$$

Given the no bequests assumption, we know that wealth during retirement years is a function of the wealth accumulated at the initial point of retirement, the rate of interest earned during retirement, retirement consumption and prices during retirement such that wealth is zero at the end of retirement (death). To find this function we employ a second differential equation, solving for the function having the derivative

$$\frac{d}{dt} w_2(t) = w_2(t) r - c_r\, p \tag{6}$$

which leads to the function, $w_2(t)$, for wealth accumulation during retirement years given the boundary condition that initial wealth at the point of retirement is w0:

$$w_2(t) = \frac{c_r\, p + e^{rt}\, r\left(-\frac{c_r\, p}{r} + w0\right)}{r} \tag{7}$$

Substituting (5) for w0 in (7), then setting (7) equal to zero (the no bequest assumption) and solving for t answers the question: how long does it take for retirement wealth to be exhausted? Essentially this determines the length of retirement.[15] This is a critical aspect of the model because the investor is presumed in (8) below to derive utility from increased retirement time. As suggested earlier, an important motive for choosing real property as an investment is the possibility of positively influencing the outcome. This model accounts for that positive influence by reducing time working for an employer, therefore increasing retirement time. Presuming that the agent has a disutility for working for an employer, the sooner retirement can occur, the better. The investor's problem now becomes:

[15] Macabre as it may seem it, in this model it also determines the date of death.

$$\max_{L,S} U\left(c_w, L, t_r\right) = \left(c_w - 900\right)^{\alpha} \left(L - .25\right)^{\beta} \left(t_r\right)^{\gamma} \qquad (8)$$

where the variables follow the notation given earlier and the constants (900 and .25) represent base minimums. These constants may be viewed as subsistence in each case so that the agent values consumption and leisure only over the minimum required to sustain life.[16]

Model simulations

Due to the lack of a pdf, the theory presented in Section 3 describing fat tails in real estate required numerical representations. The model employed in this section to explain fat *right* tails in real estate contains differential equations. As most differential equations do not admit a closed form solution, one must rely on numerical and graphical demonstrations rather than the conventional comparative statics permitted by the analytic form. The best one may do in this regard is a series of simulations. This is consistent with Markowitz' (1991b) advice regarding modeling the individual investor: "In sum, I encourage readers with requisite skills to try building and using realistic game-of-life simulators; and editors to look kindly on the publication of their results." (p.5)

Imposing certain conditions on the model such as

$$L = 1 - H - S \qquad (9)$$

and

[16] This functional form is chosen in part for convenience. The simpler, general, two period model, $U(C, L) = U(C_1, L_1) + U(C_2, L_2)$, was attempted. However, given the constraints, the second order condition for this model contains variables that must have known values to confirm that the Hessian matrix of the function is negative definite.

$$0 < \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} < 1; \alpha + \beta + \gamma = 1 \qquad (10)$$

and requiring plausible but fixed values for a number of variables, it is possible to create a graphic showing how different combinations of three of the most critical variables produce different optimum utility. The variables chosen for the graphic are not the choice variables and the graphic is not a conventional optimization exercise. Rather, it depicts ranges of optimal utility that may be achieved under different conditions.[17] The three important variables used in the simulations below are the efficiency factor (ρ), working years (t_w) and the portion of one's time devoted to real estate ownership activities (S). The intuition is that persons of varying skill levels devote different amounts of time to maximizing their real property investments while also working for an employer with the result that they retire from employment at different times in the process. Three specific simulations of the process are provided in Figures 10 – 12 below.

Table 2 shows a selection of values for certain variables in the first simulation that will serve as a baseline case.

[17] The full elaboration of this process may be found in a Mathematica notebook that is part of the electronic version of this paper, available for downloading from the author's web site.

Table 2: Values for variables in Simulation #1

Variable	Value
T	60
α	0.1
β	0.4
γ	0.5
c_r	1200
p	20
r	0.07
wg	120000
H	0.25
W_0	50000

Inserting these values into the equations above and substituting, the next step is to take the first partial derivative of the utility function with respect to S and t_w, set them equal to zero and solve for optimal utility using a number of different values of ρ, t_w, and S. Figure 10 reflects the graph of optimal utility, given the data in Table 2, showing how optimal utility changes with changes in the critical variables. Note that time working prior to retirement decreases with higher values of ρ, S or combination of them.

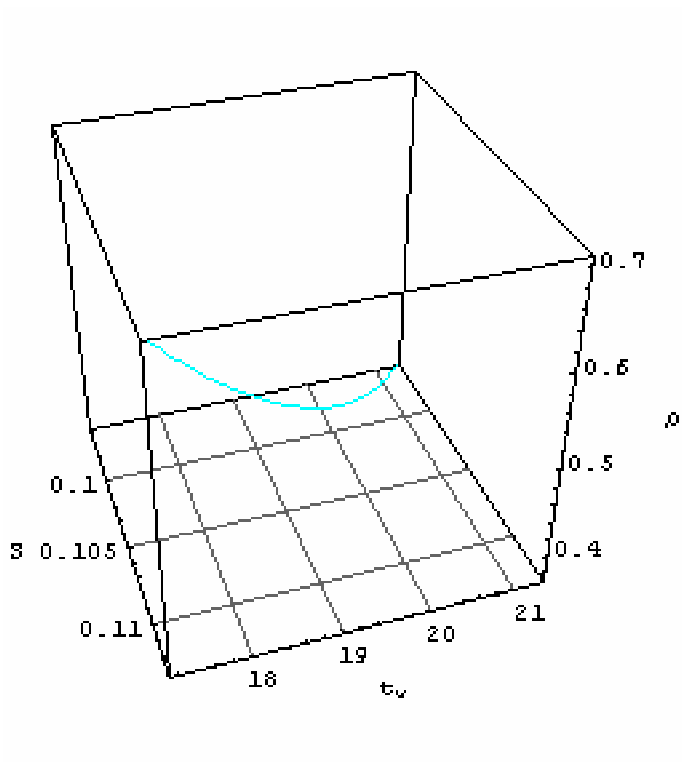

Figure 10: Simulation #1 - Plot of Optimal Utility as S, t_w and ρ change

Table 3 changes two fixed parameters for the next simulation. Increasing the value of γ and decreasing β indicates a greater preference for early retirement and a lower preference for leisure. This leads to a second simulation, shown in Figure 11, that plots consistently lower time working, therefore longer retirement times, for all values of ρ and S. Thus, like the fable of the ant and the grasshopper, the investor willing to forego more leisure early in life is able to retire sooner.

Table 3: Values for variables in Simulation #2

Variable	Value
T	60
α	0.1
β	0.2
γ	0.7
c_r	1200
p	20
r	0.07
wg	120000
H	0.25
W_0	50000

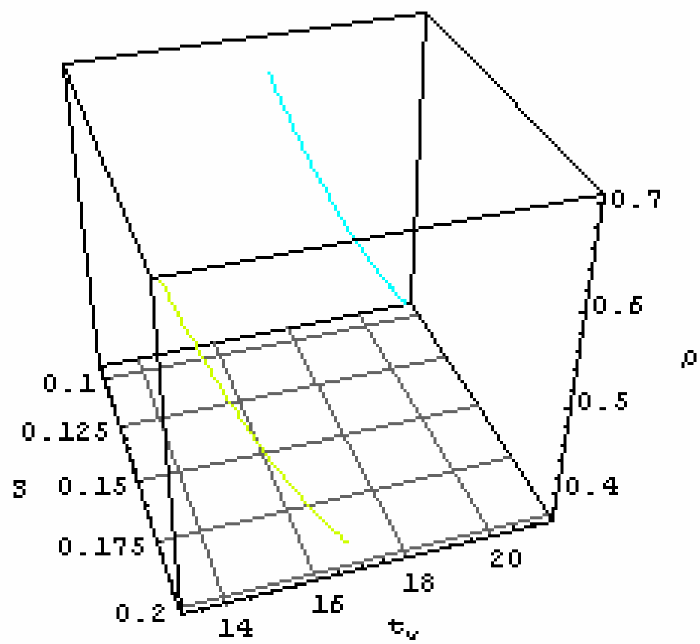

Figure 11: Simulation #2 - Optimal Utility as S, t_w and ρ change with higher γ and lower β

The number of permutations one can make is very large. Some simulations are out of the bounds of reality or produce invalid solutions such as negative time working. A final simulation concludes this exercise, demonstrating one other change. Values in Table 4 return β and γ to their values in the first simulation, however the value of H changes from .25 to .15 indicating a change in the percentage of time devoted to work for an employer.[18] Figure 12 plots a graph

[18] One can hypothesize any number of reasons for this – semi-retirement, decreased travel time, etc.

behind the two produced by Simulations #1 and #2 indicating that devoting a smaller portion of time to work for an employer, given the data in Table 4, delays retirement by requiring a longer time working at all levels of t_w and S.

Table 4: Values for certain variables in Simulation #3

Variable	Value
T	60
α	0.1
β	0.4
γ	0.5
c_r	1200
p	20
r	0.07
wg	120000
H	0.15
W_0	50000

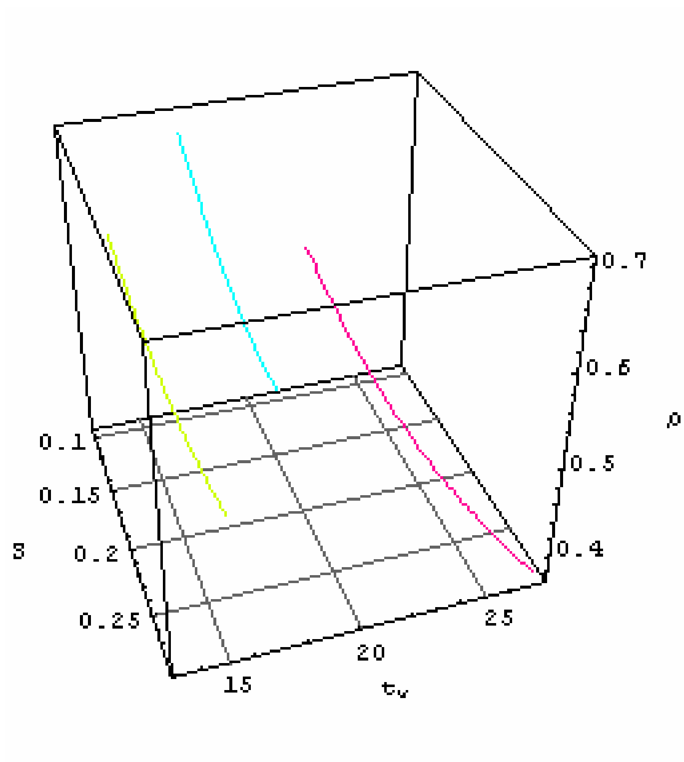

Figure 12: Simulation #3 - Plot of Optimal Utility as S, t_w and ρ change under conditions of higher H

Conclusion

If the investor adds efficient labor in order to accelerate retirement, he does so by making P_{t+1} in (1) larger than it would have otherwise been. Different agents exhibiting different levels of efficiency in real property ownership will self-select into the private real estate investment market. Theoretically, these agents will experience different right tail investment return outcomes depending on the time they devote to their property, their efficiency, the demands of their employer, the length of their life and a host of other realistic game-of-life conditions.

5. The Empirical Evidence

With a theoretical argument for heavy right tails, we visit the data for Tier II investment returns. Such data are becoming more plentiful due to the recent and on-going digitization of this market. Our interest is in the *shape* of Tier II market real estate return distributions and estimates of the parameters of the stable characteristic function.

Investment real estate data in sufficient quantity are difficult to obtain. Even when a large enough data set exists, researchers still face one of two problems. Either (1) they have confirmed sales but no intertemporal holding period income or (2) they have intertemporal income on properties that have not sold. These problems are associated, respectively, with Tier II and Tier III property. Large, publicly held institutions keep meticulous records and deliver regular reports to their constituents, making operating income information comparatively plentiful. On the other hand, these properties typically have long holding periods and infrequent sales. To compute annual returns for Tier III property, appraised values are substituted as hypothetical sales to generate equity reversions for the final cash flow. This leads to the widely discussed "smoothing" problem.[19] Data for Tier II property are quite different. Operating information is private (if it exists at all) but sales are more frequent. Here, it is common to

[19] cf. Ross and Zisler (1991). It also raises other issues in a data set when the same properties repeat in the same data set over time. Young and Graff (1995) encountered this problem and so shall I. It may be that this concern is endemic to this type of research and may never be overcome.

assume that value is a linear function of income and work with only sale information using the change in price over time as a proxy for return. That is my situation.

Source and Methodology

Source

The data are obtained for the San Diego County multifamily residential market. Included are a total of 6,537 confirmed sales (gross before elimination due to missing variables) occurring during the 183 months between March 1983 and August 1998. There are 4,514 different properties located among 81 different postal zip codes; 94.17% of all sales involved less than 100 units (6,156 of the 6,537 observations). There are 2,013 properties in the data set that sold more than once during the time period. Multiple sales of the same property are shown in *Table 5*.

Table 5: Sales Repeating in the Data

46.63%	3048	Properties appearing only once in the data base
22.58%	1476	1st appearance of a property that sold more than once
22.58%	1476	2nd appearance of a property that sold twice
6.75%	441	3rd appearance of a property that sold three times
1.28%	84	4th appearance of a property that sold four times
0.18%	12	5th appearance of a property that sold five times
100.00%	6537	Total properties in the data base
30.79%	2013	Total properties in the data base with multiple sales

Methodology

As the data lack intertemporal income, returns for the Tier II property type will be based on the change in price over time. Returns over a holding period will be calculated as if continuous, given that holding period is measured by days.[20] Thus:

$$R_t(z) = \frac{1}{n}\left[\ln\left(\frac{P_t}{P_{t-n}}\right)\right] \tag{11}$$

Where:
$R_t(z)$ = the return at the end of the holding period at time t at location z (zip code);
n = the number of days in the holding period;
P_t = the price at the end of the holding period; and
P_{t-n} = the price at the beginning of the holding period.

Interest is in how asset-specific risk varies. Hence, linear regression is employed to produce a term assumed to model this risk. This term, according to a variation on Eq. 4 of Young and Graff,[21] is the error term in the following regression equation:

$$R_t(z) = \xi + \eta_t(h(z)) + \varepsilon_t(z) \tag{12}$$

Where:
$R_t(z)$ = the return at the end of the holding period at time t at location z (zip code);
ξ = the intercept;
η = coefficient conditioning the mean for location;
h(z) = a dummy variable representing location (zip code); and
$\varepsilon_t(z)$ = disturbance term

[20] The alternative, daily compounding method, $R_t(z) = \sqrt[n]{\frac{P_t}{P_{t-n}}} - 1$, produces a trivially different result.

[21] Young and Graff used property type to condition their mean. Because all of my property is the same type, attributing non site-specific risk to location seems more appropriate.

The error term will be assumed to be the excess return rewarding investors for taking asset-specific risk. Using maximum likelihood estimation provided by Nolan (1997), this asset-specific risk, $\varepsilon_t(z)$, will be fit to the stable distribution, resulting in an estimation of α for Tier II real estate.

Descriptive Statistics

Descriptive statistics for selected variables appear in *Table 6* where

Units	=	Number of dwelling units in the building
Sale_Pr	=	Sale Price of the building
Age(Yrs)	=	Age of the building in years
Lot Size (SF)	=	Number of square feet of land
Cap Rate	=	Capitalization Rate (Net Operating Income ÷ Sale Price)
GIM	=	Gross Income Multiplier (Sale Price ÷ Gross Income)
HP	=	Holding Period in days

Table 6: Descriptive Statistics

	Units	*Sale_Pr*	*Age (Yrs)*	*Lot Size (SF)*	*Cap Rate*	*GIM*	*HP*
Mean	29	1,470,823	20	49,660	8.448	7.48	1,609
Standard Error	1	48,348	0	1,836	0.024	0.03	24
Median	12	575,000	18	13,210	8.179	7.75	1,366
Mode	8	450,000	20	6,969	7.920	8.33	394
Standard Deviation	53	3,869,349	15	148,209	1.906	2.01	1,061
Sample Variance	2767	1.50E+13	229	21,965,948,458	3.632	4.05	1.13E+06
Kurtosis	63	367	3	533	14.182	0.25	0
Skewness	6	14	1	16	2.218	0.05	1
Range	1065	126,915,000	93	6,264,799	32.083	15.21	5,157
Minimum	5	85,000	-	871	0.003	1.69	16
Maximum	1070	127,000,000	92	6,265,670	32.086	16.90	5,173
Sum	189930	9,420,619,689		323,586,695			
Count	6537	6,405	5,823	6,516	6,154	4615	2,013
Confidence Level(95.0%)	1	94,778	0	3,599	0.048	0.06	46

Descriptive statistics for those properties appearing in the data set as repeat sales are shown in *Table 7*.

Table 7: Description statistics for continuous returns on property sales repeating in the data

2013 Repeat Sale Continuous Returns	
Mean	0.000078
Standard Error	0.000011
Median	0.000044
Mode	0.000000
Standard Deviation	0.000476
Sample Variance	0.000000
Kurtosis	33.570104
Skewness	3.615507
Range	0.008829
Minimum	-0.002800
Maximum	0.006029
Sum	0.156630
Count	2013
Largest(1)	0.006029
Smallest(1)	-0.002800
Confidence Level(95.0%)	0.000021

Tests performed

The primary tests are those that point to the conclusion that the population from which the sample is drawn is SP distributed. Unfortunately, it is not possible to *prove* that a data set is stable (Nolan (1998a)). Rather, diagnostics are used to detect non-stability.[22] The two most reliable of these diagnostics are graphical.

First, prepare and inspect graphical forms of the data. This can be done in several ways.

[22] Although this may sound troubling, Nolan (1998a) reminds us that "testing for normality is still an active field of research" (p.11).

1. Prepare a histogram of the data and compare it with the shape of a similarly scaled and located normal distribution.
2. Generate a set of points from the random generating methods provided by Nolan (1997) using the parameters estimated from the data, then plot those points to obtain a smoothed plot similar to the histogram.[23]
3. Finally, using the transform techniques described in Brown (2000), one may create a plot of the simulated pdf for the data, comparing that with a normal distribution. If the data are SP distributed, all should result in the same outcome: Long, heavy tails support a conclusion of stability.

A second diagnostic involves constructing a stabilized probability plot (Michael (1983)) using STABLE.EXE software developed for that purpose (Nolan (1997)).[24] If the distribution is stable, such a plot, when compared to a diagonal line through the origin, should minimize deviations from the line.

If graphical tests are consistent with the hypothesis of stability, the next step is to fit the data to a stable distribution, estimating parameters using STABLE.EXE then computing confidence intervals according to Nolan (1998a).

[23] The "chicken-and-egg" notion that I am using parameters estimated later to generate these graphics now is not ignored. The processing of seeking stability in a distribution and a sample from it involves interrelated methods, the starting point of which is in the hands of the analyst.

[24] This software may be obtained from: http://www.cas.american.edu/~jpnolan/stable.html

Results

Graph of the data

As discussed above, three different graphical demonstrations will be made to detect non-stability. *Figure 13* shows a histogram of the residuals for 2,013 repeat sales. Adopting a variation on the Young and Graff (1995) methodology, the regression that produced these residuals has removed the location effect. Thus, these are assumed to be the returns to *site-specific* risk encountered by individual real estate owners. In the background is the normal distribution. While not dramatic, there is evidence of long tails, slightly heavier on the right.

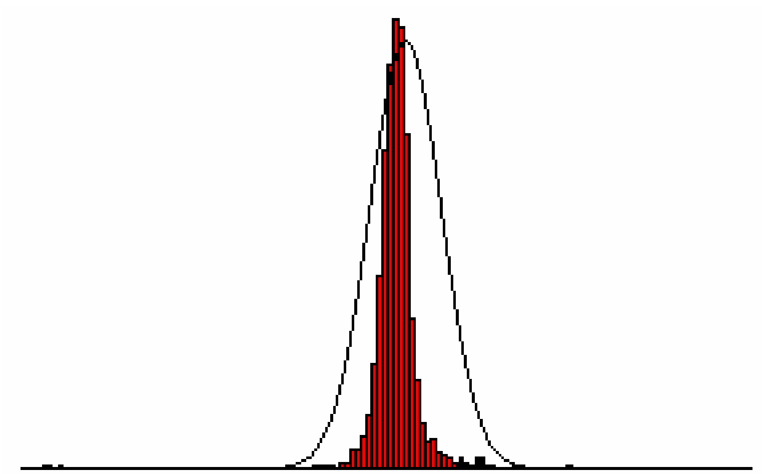

Figure 13:Histogram of the data with the normal distribution in the background

Figure 14 is a plot of a random sample generated with the parameter values developed below. This also shows a heavy tail skewed right.

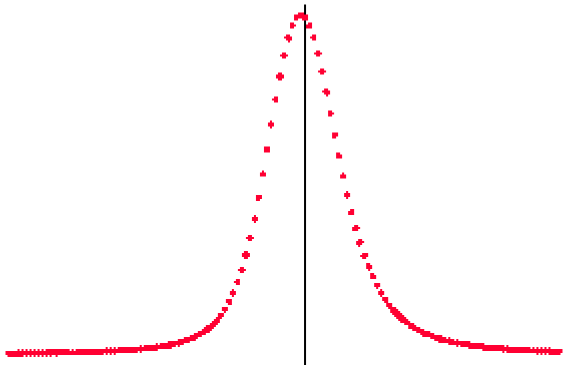

Figure 14: Smoothed plot of a distribution of random numbers generated using parameters estimated for the data

Finally, *Figure 15* shows a plot of the "simulated" pdf constructed using the inverse Fourier transform methods of Brown (2000). The parameters, S^0(1.4218, .2579, .000171215, -.0000594559), estimated below by the method of Nolan (1997), are used to generate this plot. The normal, as before, shadows the distribution for the sake of comparison. Note the additional variation in the tails similar to *Figure 4* and the shift in variation from the left to the right tail similar to *Figure 5*.

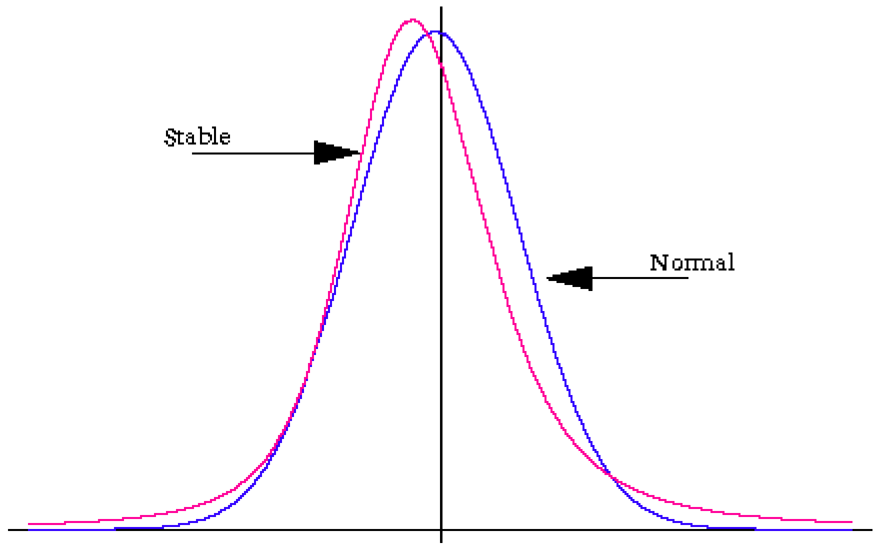

Figure 15: "Simulated" stable pdf created by evaluating the Inverse Fourier Transform of the characteristic function

Stabilized Probability Plot

As suggested by Nolan (1998a), the Stabilized Probability Plot (Michael, 1983) is constructed and appears as *Figure 16*. Evidence that the data are stable is found when the plot of the points deviates very little from a diagonal line through the origin. *Figure 16* has such an appearance, further supporting the hypothesis of stability.

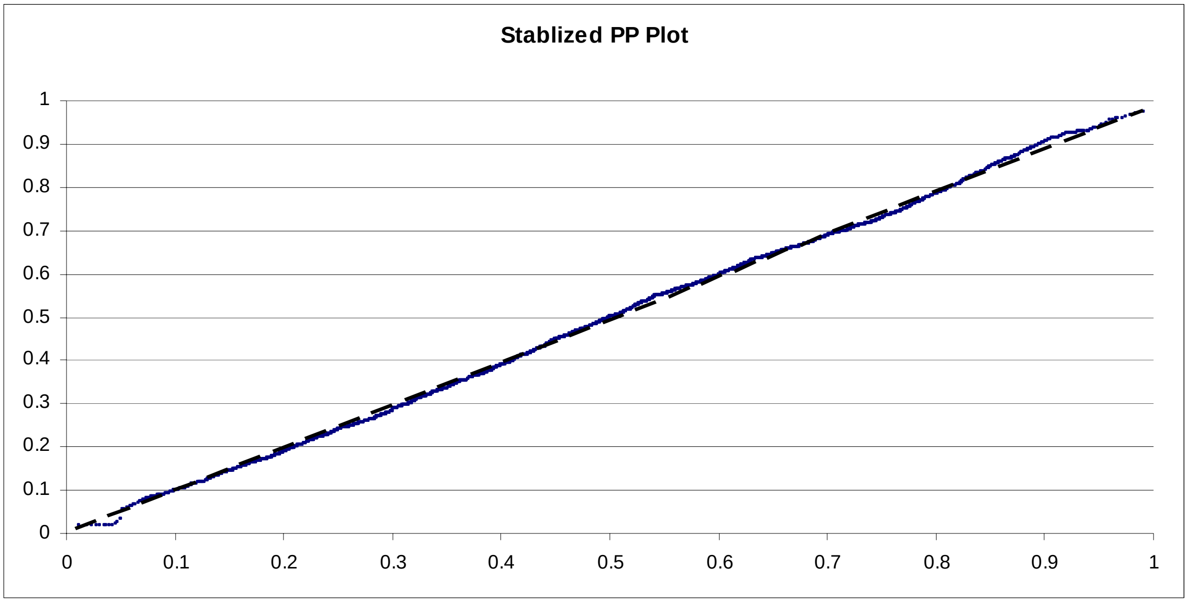

Figure 16: Stabilized Probability Plot

Fit of the Data

The hypothesis of stability having graphical support, the data are fit to a stable distribution using STABLE.EXE provided by Nolan (1997). *Table 8* shows parameter estimates and confidence intervals for the 90% and 95% levels for α and β.[25]

Table 8:Confidence intervals for estimates of α and β

Conf. Level	**n**	**α**	**β**	**CI_α**	**High**	**Low**	**CI_β**	**High**	**Low**
90%	**2013**	**1.4218**	**0.2579**	**0.0571**	**1.4789**	**1.3647**	**0.0951**	**0.3530**	**0.1628**
95%	**2013**	**1.4218**	**0.2579**	**0.0681**	**1.4899**	**1.3537**	**0.1133**	**0.3712**	**0.1446**

Confidence intervals are computed from the standard deviation of the parameter based on tables from Nolan (1998a) according to the following formula:

[25] STABLE.EXE also provides estimates of γ (.000171215) and δ (-.0000594559) which are ignored at this time.

$$\hat{\theta}_i \pm Z_{\alpha/2} \frac{\sigma_{\hat{\theta}_i}}{\sqrt{n}}$$

where

$\hat{\theta}_i$ is the estimated individual parameter of the parameter vector θ

$Z_{\alpha/2}$ is the two-tailed level of confidence desired from the standard z-table and

$\sigma_{\hat{\theta}_i}$ is the standard deviation of the individual parameter from the table given in Nolan (1998).

As usual, confidence intervals are an artifact of the sample size. As n grows larger, the Z-table adjustment to the estimated individual parameter becomes smaller.

6. Conclusion

At a minimum, the proper application of Modern Portfolio Theory requires perfect divisibility, perfect liquidity and perfect reversibility, three properties that virtually everyone agrees is lacking in the real estate market. The further requirement that returns be normally distributed essentially ends any possibility of applying MPT to individual real estate investing. Lusht (1988) had it right when he said, "now that we have confirmed what doesn't work, we should stop [using the static equilibrium models of mainstream finance]" (p. 102). Yet another finding of heavy tails in real estate return data, this time in the Tier II market, adds to the suspicion that MPT is not for real estate.

Anecdotal evidence has long suggested that the market for private real estate investment is populated in part by entrepreneurs who earn "sweat equity" while moonlighting from their day jobs. This paper attempts to formalize some of their motives, explain their behavior and demonstrate the outcomes of that behavior in the form of return distributions observed in that market. Knowing that they cannot easily diversify away site-specific risk, one wonders if they seek such risk in order to employ their abilities to maximize the opportunities inherent therein. For these investors the Laundry List becomes a set of opportunities rather than a set of burdens.

The natural extension of this work is to find out if these results can be reproduced. With the tools in place and increased data sources for Tier II property it should be a simple matter to determine if the results here are persistent in other cities and across different time periods.

One of the most significant institutional factors affecting real estate is taxation. Of course this benefit is priced along with all others and may not show up in the form of larger or smaller returns. But Tier II owners who add labor, convert their time into equity and do so on a non-taxable basis as long as they do not sell the property.[26] This may be seen to affect behavior and influence the decision to acquire real property rather than financial assets as well as the decision to not to commit more time to ones (currently taxable) employment.

There is a growing body of research on the left side of the heavy tail distribution known as Extreme Value Theory (EVT). Primarily interested in the downside of risk, many international financial intermediaries base loss control policies on EVT. But the glass is also half full. The heaviest of the heavy right tails may be found in the field of real estate development, where there is virtually no theory and little data. Combining EVT and real estate in this arena could produce the first theoretical and empirical results for land developers.

If empirical support for the theory presented here grows, future research can investigate other aspects of the asset specific risk investors assume in this market. One such inquiry would be to add financing to the return data and attempt to discover if left tail observations are associated with leverage. Research opportunities abound in this little studied market

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[26] Under IRC Section 1031 they also have an option to tax defer capital gains inherent in that equity.

The real estate market is heavily affected by its ownership. Due to the separation of ownership and control, the return on financial assets – including Tier III where real estate is essentially converted into a financial asset – is less influenced by ownership. The private real estate investment market may be the opposite. Ownership and control are usually concentrated in the same party. There are labor issues, agency cost savings, and entrepreneurship matters influencing returns in this market. In short, a share of stock does not care who its owner is.[27] Perhaps in some way an apartment building does.

[27] Unless it is part of a controlling block of shares, the exception rather than the rule.

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