# Macroeconomic Conditions, Systematic Risk Factors, and the Time Series Dynamics of Commercial Mortgage Credit Risk *

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# Macroeconomic Conditions, Systematic Risk Factors, and the Time Series Dynamics of Commercial Mortgage Credit Risk

**Abstract**

I study the time series dynamics of commercial mortgage credit risk and the unobservable systematic risk factors underlying those dynamics. A first-passage model with equilibrium macroeconomic dynamics is presented, and the default hazard rate is solved. The solutions are then put into a state space form and estimated with real world commercial mortgage performance data using extended Kalman filter. Results show large variations of credit risk over time in the commercial mortgage market, and that these variations are well explained by two mean-reverting latent risk factors. One is the macroeconomic factor and the other is a commercial property market-specific factor. The model and the results will be useful in default risk prediction, hedging and pricing.

*Keywords:* Credit risk, commercial mortgage, first-passage model, default hazard rate, Cox proportional hazard model, state space model, extended Kalman filter.

# 1 Introduction

Credit risk not only varies across different firms or borrowers, but also changes over time depending on economic environments. Understanding the time series dynamics of credit risk is equally, if not more, important than knowing the cross sectional variations in credit risk. For example, portfolio risk managers want to have the ability of distinguishing "good" times from "bad" times in addition to telling "good" assets from "bad" assets; regulators and policy makers desire to know how fluctuations of credit risk in the financial markets are related to changes in macroeconomic conditions and whether there are credit cycles; and financial economists seek to incorporate the dynamics of systematic risk factors into their credit risk pricing and measurement models[1].

In this paper, I study the time series dynamics of commercial mortgage credit risk and the unobservable systematic risk factors underlying those dynamics. The research tasks are carried out within a structural model framework. Given the incomes of commercial properties sustaining commercial mortgages are largely affected by the macro economy, I modify the first-passage model of Black and Cox (1976) and Longstaff and Schwartz (1995) to incorporate equilibrium macroeconomic dynamics. Aggregate economic output and commercial property cash flows are the primitive processes in my model. The economy has some business cycle characteristics, as modeled in Goldstein and Zapatero (1996), in that the expected growth rate of aggregate output is mean-reverting. I model the commercial property cash flow as determined by two unobservable variables, the state of the economy and a commercial property market-specific factor, similar to the approach by Tang and Yan (2006) in modelling firm cash flow. Mortgage default occurs when the cash flow of the underlying property falls below a default boundary. I solve the first-passage time density and default hazard rate of a representative commercial mortgage, which gives the relationship between mortgage credit risk and the two unobservable state variables.

My empirical work takes advantage of a large commercial mortgage performance dataset available and focuses on default probabilities. The main objective is to extract information from real world default experience about the latent systematic factors that drive the evolutions of commercial mortgage credit risk. I first use a Cox proportional hazard model to control the heterogeneity of commercial mortgage loans and to estimate the default hazard rate time series of a representative mortgage. The hazard rate time series are then used to estimate the first-passage model in state space form, in which the risk free bond yield and the mortgage default hazard rate are the two variables treated as observed. I estimate the nonlinear state space model with extended Kalman filter. Parameter estimates reveal the relationships between default probabilities and the risk factors, and state variable estimates

[1]See Wilson (1998), Jarrow and Turnbull (2000), Allen and Saunders (2003), Koopman and Lucas (2005), and Duffie, Wang and Saita (2006) for elaborations.

capture the dynamics of the unobservable systematic risk factors.

Empirical results show substantial variations in commercial mortgage credit risk over time. The default hazard rate in "bad" years is 15 times as high as that in "good" years. However, the variations are largely explained by the dynamics of the two mean-reverting unobservable risk factors. The expected economic growth is a significant factor with long term mean of 1.70%, volatility of 0.63% and annual mean-reversion speed of 0.46. In addition, there is a significant commercial property market-specific factor with long term growth rate of 1.20%, volatility of 0.6% and mean-reversion speed of 0.60. Changes of these two factors explain over 85 percent of the variations in default hazard rates. Simulations show that commercial mortgage default risk becomes more sensitive when the economy turns down. Based on the model estimates, the long term mean of annual default hazard rate is 0.06% for a newly originated loan and 2.58% for a five-year seasoned loan. Several other findings verify the conventional wisdom. For example, empirical estimates reveal that the insolvency threshold for default is 0.89 assuming an underwriting debt-service-coverage ratio (DSCR) of 1.3. This is consistent with the notion that mortgage borrowers do not default ruthlessly possibly because of high transaction cost. Moreover, significant seasoning effect has been found. My estimates show that default hazard rate reaches its peak after four and a half years of loan origination.

The main contribution of this research is to exploit the time series perspective of credit risk in the commercial mortgage market. This is largely motivated by recent discussions of dynamic credit risk modelling. For example, Wilson (1998) stresses the importance of understanding how credit risk changes over business cycles. Jarrow and Turnbull (2000) argue for incorporating dynamics of "common economic influence" into credit risk pricing models. Allen and Saunders (2003) suggest capital regulators to consider the banking pro-cyclicality. Recently, Koopman and Lucas (2005) and Pesaran, et al (2005) have developed time series models to study credit risk and macroeconomic dynamics. Duffie, Wang and Saita (2006) estimate corporate default risk dynamics by incorporating autoregressive macroeconomic and firm-specific covariates. My research follows this line of thinking, but uses a structural model to examine the time series dynamics of credit risk. The estimated model is useful in predicting long term dynamics of credit risk. It also provides hedging implications for portfolio credit risk. Moreover, pricing of commercial mortgages can be conducted with the estimated default probabilities and the transformation of real measure into risk neutral measure implied by my structural model.

The mortgage default risk literature has long incorporated interest rate and property market growth rate as systematic factors in the mortgage valuation and default prediction models (See, e.g. Kau et al 1987, Titman and Torous 1989, and Deng, Quigley and Van Order 2000). However, the emphasis has not been put on the time series properties of default risk. Further, there is a gap between theoretical default risk models and empirical default estimations in treating the systematic risk factors. On one hand, theoretical pricing models incorporate the dynamics of these risk factors, but on the other hand empirical estimations only apply reduced-form approach to put ex post realizations of these factors into the model and thus miss the dynamics of these factors. Here I bridge the gap by solving a structural model and then estimate the structural model in state space form. The dynamics of the risk factors is estimated within my integrated model.

Another important feature of the current study is that it studies unobservable risk factors. Recent research has found that there are a small number of latent variables playing fundamental roles in the economy, and that many of the observable variables are determined by these latent factors (Bai and Ng 2004). In fact, economic theories have long found it useful to explain observed economic data by some fundamental unobservable factors [2].

My empirical approach is similar to that of Duffee (1999), who uses extended Kalman Filter to estimate the default intensity process of corporate debt from corporate and treasury bond yields. The main difference is that Duffee (1999) uses a reduced-form credit risk pricing model, which directly assumes the default intensity to follow an exogenously given process. I pay more attention to the economics of default by using a structural model.

While the research mainly helps us understand the long term dynamics of systematic credit risk, it also considers the interactive impact of idiosyncratic risk factors and systematic risk factors on individual commercial mortgage loan default. The Cox proportional hazard model used in the study helps identify the effects of loan specific characteristics on default. For example, estimates show substantial variations in default hazard rate across geographic regions; maturity and amortization terms are found to be negatively correlated with default hazard rate; and certain property types such as hotel and healthcare show significantly elevated default risk. These estimates add to the existing literature additional information on cross sectional properties of default risk. They are useful in industrial applications.

The rest of the paper is organized as follows: the next section reviews existing literature closely related to my research. Section 3 presents the first-passage model. Section 4 describes my empirical methodologies, and section 5 reports data and results. Conclusions and discussions are in a final section.

[2]For example, the arbitrage pricing theory (APT) of Ross (1976) assumes the existence of a set of theoretically unidentified common factors underlying all asset returns. Most term structure models of interest rate including Vasicek (1977) and Cox, Ingsoll and Ross (1985) assume an unobservable "short" rate. In fact, Stambaugh (1988) derives the conditions under which an affine term structure model implies a latent-variable structure for bond returns.

## 2 Related Literature

My research questions are largely motivated by recent discussions in the literature about the time series dynamics of credit risk in corporate debt and bank loans. I merge the two branches of credit risk literature, credit risk pricing studies and default prediction studies, in forming an integrated default risk model.

### 2.1 Credit Risk Dynamics and Credit Cycles

How credit risk changes over time has been an important topic in corporate credit risk literature. Early research focuses on the determinants of default rates and credit spreads. For example, Altman (1990) examines the relationship between changes of business failure rates and changes of macroeconomic variables such as GNP, money supply and market index. Longstaff and Schwartz (1995) and Duffee (1998) study how changes in treasury yields affect changes in corporate credit spreads.

Later on, researchers explore the characteristics of credit risk changes, especially the cyclical patterns. For example, Carey (1998) documents significant differences in default rates for "good" years, as compared to "bad" years. Wilson (1997, 1998) finds cyclical default rates and stresses the importance to develop models that account for changes of default risk over the business cycle [3]. Allen and Saunders (2003) emphasize the importance of pro-cyclicality in the banking system and argue that capital regulations must take into account the cyclical fluctuations in credit risk to avoid amplification of the pro-cyclicality. Erlenmaier and Gersbach (2001) find both the levels and standard deviations of default rates vary throughout the business cycle.

Based on these observations, researchers have recently developed models to try to capture the time series dynamics of credit risk. Gordy (2003) presents a theoretical risk-factor model for bank loan default risk in the spirit of the APT model of Ross (1976). Koopman and Lucas (2005) and Koopman, Lucas and Klaassen (2003) apply the unobserved components model to empirically estimate default cycles. They find evidence of 10 year cycles in US business failures and cyclical co-movements between GDP and business failures. Pesaran, et al (2005) develop a vector autoregressive model to study credit risk profile of commercial banks and macroeconomic dynamics. Hose and Vogl (2005) directly estimate an autoregressive model with exogenous macroeconomic input for default probabilities. Duffie, Wang and Saita (2006) emphasize the needs of a dynamic approach in order to make multi-period predictions of corporate default. They estimate conditional probabilities of corporate default, incorporating autoregressive macroeconomic and firm-specific covariates, and show

[3] Even before that, Blume, Keim and Patel (1991) and Blume and Keim (1991) conjecture that the seasoning effect found in corporate default might be a manifestation of business cycle effect.

that their model outperforms other static models in out-of-sample predictions.

The mortgage default risk literature has long incorporated systematic risk factors in the models. Interest rate and property value growth rate are two widely used variables. However, the emphasis is placed on either the ability of the pricing models to explain observed credit premium (See, for example, Kau et al 1987, Titman and Torous 1989, Schwartz and Torous 1992, and Childs, Ott and Riddiough 1996) or the cross sectional differences in default behavior (See, for example, Vandell et al 1993, Deng, Quigley and Van Order 2000 and Ambrose and Sanders 2002). A few exception are Gabriel, Rothberg and Nothaft (1989), Maxam and Fisher (2001) and Riddiough (2004), which use simple regression models to study the determinants of MBS (CMBS) credit premium. Therefore, the time series properties of default risk are under-studied, especially from a dynamic perspective.

## 2.2 Credit Risk Pricing

A vast literature has been developed for credit risk pricing, which includes two strands of models. The structural models emphasize the reason why firms default. These models usually start with a specification of asset value process, and assume debt holders default when asset values fall under certain default thresholds. The classic Merton (1974) model specifies a mean-reversion asset value process and assumes interest rate is constant. The debt value is just the asset value minus the equity value, which is similar to a call option. Applying the Merton-Black-Scholes option pricing technique, the equity value can be easily found out, and thus the debt value is determined.

The Merton (1974) model assumes that default only happens at maturity. Black and Cox (1976) relax this assumption and introduced the first-passage model, in which default happens at the first time the asset value falls under a boundary. Given an asset value process and a default boundary, it is easy to find out the default probabilities at each point in time, as well as the debt payment outcome at various states. The debt value is just the expected discount value of the payments under risk neutral measure. They assume a pre-specified time varying boundary and give the closed-form solution of debt value. Later on, Longstaff and Schwartz (1995) generalize the first-passage model to allow stochastic interest rate. Assuming an exogenously given constant default boundary, they give closed-form formulas for fixed rate and variable rate debt. Relaxation of maturity-only default is very important because in real world, corporate debt, bank loans and mortgages can default at any time before maturity [4].

In a structural model, default probabilities are endogenously determined within the

[4]Important extensions of the structural model include Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996), Duffie and Lando (2001), Collin-Dufresne and Goldstein (2001) and Huang and Huang (2003). See Duffie and Singleton (2003) for more discussions.

model . In contrast, the reduced form approach directly assumes that there exists an exogenously given default intensity. For example, Jarrow and Turnbull (1995) assume default follows a Poission process to arrive at a constant rate at each point in time [5]. The advantage of this approach is that it greatly simplifies the debt valuation. Lando (1998) shows that with this approach, the zero-recovery defaultable bond price is just the expected discount payoff with a discount rate of $r + \lambda$ instead of the risk free rate $r$, where $\lambda$ is the default intensity under risk neutral measure [6]. Duffee (1999) empirical studies the ability of the reduced model in fitting observed corporate credit spreads. Apparently the reduced form models pay less attention to the economics of why firms or borrowers default.

The mortgage credit risk valuation literature generally follows the Merton (1974) model. For example, Cunningham and Hendershott (1984) value mortgage default insurance using the contingent claims pricing approach. Assuming a diffusion process for house value, they construct the Merton-Black-Scholes pricing PDE, specify the boundary conditions, and use numerical methods to obtain the fair premiums for different FHA insurance contracts. Kau et al (1987) generalize the model to have two factors, the stochastic property value and risk-free interest rate. Titman and Torous (1989) and Schwartz and Torous (1992) take the same approach to study whether the model generates default premiums consistent with those observed in the marketplace. Further generalizations include Giliberto and Ling (1992), Childs, Otts and Riddiough (1996), Capozza, Kazarian and Thomson (1998), and Downing, Stanton and Wallace (2005), among many others.

Recently, Kau, Keenan and Smurov (2006) apply the reduced-from model of credit risk to value residential mortgages with default (and prepayment) risks. They assume an exogenously given default intensity and use particle filter to estimate the process from mortgage performance data, and then calibrate their model to observed mortgage rates [7] [8].

My study builds on the first-passage structural model of Black and Cox (1976) and Longstaff and Schwartz (1995). Different from most structural models that use a calibration approach for empirical investigations, I follow Duffee (1999) to empirically estimate my structural model using extended Kalman filter.

[5] Equivalent to a constant hazard rate and an exponentially distributed duration time.

[6] Other generalizations include Mandan and Unal (1996), Jarrow, Lando and Turnbull (1997), Duffie and Singleton (1998), Jarrow and Turnbull (2000) and Mandan and Unal (2000).

[7] One difficulty of this approach is the adjustment of the real default probabilities estimated from default data into risk neutral probabilities for pricing.

[8] In fact, Riddiough and Thompson (1993) take the similar approach of specifying exogenously given default probabilities for their pricing model, but they use deterministic rather than stochastic functions.

## 2.3 Default Prediction

Predicting default probabilities is important. On one hand, value of the debt is determined by the probability of default together with loss given default. On the other hand, creditors and rating agencies make decisions of loan origination and bond rating directly based on predicted default probabilities.

Early research such as Altman (1968) and Ohlson (1980) use dscriminant analysis and probit model to form the "Z-score" and "O-score" based on accounting variables, which later become widely accepted financial distress measures. Lennox (1999) compares discrete choice models with the discriminant analysis and conclude that a well-specified probit or logit model has better performance. Campbell, Hischer and Szilagyi (2006) also apply a logit model to construct a measure of financial distress to study stock returns.

Most recent research on bankruptcy predictions uses duration analysis. For example, Lee and Urrutia (1996) use a duration model based on a Weibull distribution of failure time. McDonald and Van de Gucht (1999) apply a Cox proportional hazard model with time varying covariates. Shumway (2001) also estimates a hazard model of bankruptcy with both accounting ratios and "market-driven" variables and finds the time-varying "market-driven" variables to play important roles in bankruptcy predictions. He also argues that the hazard model produces more accurate predictions than does the static logit model [9].

Mortgage default prediction studies have been explosive in the past decades possibly because of the wide availability of mortgage performance data. Early studies apply linear regression or logit models to estimate default determinants, and only use snapshot variables such as borrower characteristics, origination loan-to-value ratio (LTV) or contemporaneous LTV, etc. (See von Furstenberg 1969, Campbell and Dietrich 1983, Foster and Van Order 1984, Vandell and Thibodeau 1993 and Archer et al 2002 among many others). Later on, researchers use proportional hazard model and logit models with event-history data to account for the path dependence of default behaviors (For example, Vandell et al 1993, Quigley and Van Order 1995, Deng, Quigley and Van Order 2000, Van Order and Zorn 2001, Clapp et al 2001, Ambrose and Sanders 2002, and An, Clapp and Deng 2005). The hazard model typically assumes the duration of a mortgage follows certain distribution and models the hazard rate (conditional default probability in discrete time) as depending on some risk factors. Empirical estimates give relationship between default hazard rate and the risk factors.

Recent advancements in mortgage default predictions include competing risks hazard models and models accounting for unobserved heterogeneity (Deng, Quigley and Van Order 2000, Deng and Quigley 2002, Deng, Pavlov and Yang 2005, Clapp, Deng and An 2005).

It is noteworthy that the hazard model and logit models with event-history approach

[9] Many more studies are reviewed in Altman, Resti and Sironi (2003) and Duffie, Wang and Saita (2006).

take a dynamic approach that acknowledges the changes of default risk over time. However, the emphasis is still on cross sectional differences of default risk. Further, the dynamics of risk factors are not integrated into the reduced-form econometric models.

# 3 A First-Passage Model for Commercial Mortgage Credit Risk

A commercial mortgage is a debt with income producing property(ies) such as retail space, office, hotel or multifamily building as collateral. The mortgage borrower is obligated to make monthly repayment of principal and interest [10]. However, in each month the borrower may choose to default, usually because of the inability to make the monthly payment. A first-passage model is ideal to study commercial mortgage default.

I generally follow the fist-passage model of Longstaff and Schwartz (1995). An important feature of commercial mortgage is that the borrower mainly relies on the net operating income (NOI) from the underlying property to make monthly mortgage payment. Therefore, I directly model the cash flow process, rather than the value process of the property. Default occurs when the NOI of the underlying property falls under an exogenously specified default boundary.

The income of a commercial property usually comes from rents paid by retail tenants or firms renting the office space. It is closely related to the macroeconomic conditions. Therefore, I model the property NOI as depending on the state of the economy and I start with an equilibrium characterization of the economy.

## 3.1 The Economy

Consider a continuous-time version of a Lucas (1978) type pure exchange economy studied by Goldstein and Zapatero (1996). The single productive technology is described by an aggregate output process

$$\frac{de_t}{e_t} = \mu_t dt + \sigma_e dW_t^e \tag{1}$$

and there is a risky security (stock) whose dividend comes exactly from the above aggregate output. The drift of the aggregate output is stochastic and follows an Ornstein-Uhlenbeck process

$$d\mu_t = \kappa(\overline{\mu} - \mu_t)dt + \sigma_\mu dW_t^e \tag{2}$$

[10] Based on pre-specified mortgage terms in the contract such as coupon rate and amortization. Some loans are interest only during certain periods.

where $\kappa$ is a positive constant representing the mean-reversion rate of the drift towards the long-term mean $\overline{\mu}$. $\sigma_e$ and $\sigma_\mu$ are positive constants for volatilities of aggregate output and its drift. $W_t^e$ is a standard Brownian motion under real measure.

Intuitively, we can think of $\frac{de_t}{e_t}$ as the GDP growth rate we observe. Given the noise, however, we cannot observe the expected growth rate $\mu_t$. Notice that in this setting, a shock affects both the realized return in the current period and expected return in the next period. This is consistent with what happens in the real world, e.g. when there is over-investment, it takes time for people to make corrections. Further, the economy has some business cycle characteristics, in that the expected growth rate is mean-reverting. The agent can invest in the stock and gain both appreciation of the stock value and the dividend paid by the stock. Denote the price of the stock by $S_t$, then the return of the stock is given by

$$\frac{dG_t}{S_t} = \frac{dS_t}{S_t} + \frac{e_t}{S_t}dt \tag{3}$$

There is also a (locally) risk-free bond accessible to the agent with zero net supply. The bond price $B_t$ at time $t$ follows the following process

$$dB_t = B_t r_t dt; \qquad B_0 = 1 \tag{4}$$

where $r_t$ is the instantaneous risk-free rate, which is determined endogenously as an equilibrium outcome.

In this economy, there is a single representative agent who is a price taker and has a power utility (CRRA) over consumption with the relative risk aversion parameter $\gamma$.

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \tag{5}$$

The agent's problem is to choose the optimal consumption and investment to maximize his expected life-time utility

$$\max_{\{c_t,\pi_t\}} E\left\{\int_0^T \exp(-\delta t)\frac{c_t^{1-\gamma}}{1-\gamma}dt\right\} \quad s.t. \quad dX_t = \pi_t\frac{dG_t}{S_t} + (X_t - \pi_t)\frac{dB_t}{B_t} - c_t dt \tag{6}$$

Here, $\pi_t$ is the amount invested in the stock, $X_t$ represents the wealth process of the representative agent, and $\delta$ is the subjective discount rate.

The equilibrium of the economy is characterized by

$$c_t^* = e_t \quad \forall \quad t \in [0, T] \tag{7}$$

As shown in Goldstein and Zapatero (1996), the equilibrium instantaneous risk-free interest rate satisfies

$$r_t = \delta + \gamma\mu_t - \frac{1}{2}\gamma(1+\gamma)\sigma_e^2 \tag{8}$$

or

$$dr_t = \kappa(\overline{r} - r_t)dt + \gamma\sigma_\mu dW_t^e$$

where

$$\overline{r} = \gamma\overline{\mu} + \delta - \frac{1}{2}\gamma(1+\gamma)\sigma_e^2$$

which is the well-known Vasicek (1977) spot rate process. The market price of risk is constant

$$\theta : \frac{\mu_t - r_t}{\sigma_e} = \gamma\sigma_e \tag{9}$$

Again, following Vasicek (1977) and Goldstein and Zapatero (1996), the equilibrium price of risk-free discount bond is

$$P(t,T,r_t) = E^Q\left\{\exp\left(-\int_t^T r_s ds\right)\middle| \mathcal{F}_t\right\} = \exp\left\{A(t,T) - B_\kappa(t,T)r_t\right\} \tag{10}$$

where

$$\begin{aligned}
A(t,T) &= \left[-\overline{r} + \frac{1}{2}\gamma^2\left(\frac{2\sigma_e\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)\right](T-t) \\
&\quad + \left[\overline{r} - \gamma^2\left(\frac{\sigma_e\sigma_\mu}{\kappa} + \frac{\sigma_\mu^2}{\kappa^2}\right)\right]B_\kappa(t,T) - \frac{1}{2}\gamma^2\frac{\sigma_\mu^2}{\kappa^2}B_2\kappa(t,T) \\
B_\kappa(t,T) &= \frac{1 - \exp\{-\kappa(T-t)\}}{\kappa}
\end{aligned}$$

It can be easily shown that after some simple formula manipulation, the risk-free discount bond yield depends on the state variable $\mu_t$ as

$$\begin{aligned}
R(t,T,\mu_t) &= -\frac{1}{T-t}\log P(t,T,\mu_t) \\
&= \frac{\gamma}{T-t}B_\kappa(t,T)\mu_t - \frac{1}{T-t}\left[A(t,T) - \delta B_\kappa(t,T) + \frac{\gamma(\gamma+1)}{2}\sigma_e^2 B_\kappa(t,T)\right]
\end{aligned} \tag{11}$$

### 3.2 The Income Producing Property

The NOI of the commercial property also depends on certain conditions specific to the commercial property market, e.g. whether there is over-supply of commercial spaces, whether the operating expenses systematically elevates during certain periods. For parsimony purposes, I assume all these effects can be summarized by a single factor independent of the macroeconomic factor. Therefore, the NOI of the income producing property is determined by the state of the economy $\mu_t$ and a commercial property market-specific factor

$\xi_t$ [11]. This approach is similar to Tang and Yan (2006) in modelling firm cash flow.

$$\frac{dK_t}{K_t} = (\beta\mu_t + \xi_t)dt + \sigma_K \rho dW_t^e + \sigma_K \sqrt{1-\rho^2} dW_t^K \tag{12}$$

The drift $\eta_t = \beta\mu_t + \xi_t$ is the expected NOI growth rate, which is composed of the aggregate economy expected growth rate $\mu_t$ and a property market-specific expected growth rate $\xi_t$ independent of $\mu_t$. $\beta < 1$ is the correlation coefficient of expected commercial property market growth rate and aggregate economy expected growth rate. $\sigma_K$ is a positive constant representing the volatility of the property cash flow. $\rho$ is the correlation coefficient between the property cash flow process and the aggregate output process. As shown in Tang and Yan, $\beta = \rho\frac{\sigma_K}{\sigma_e}$. $W_t^K$ is a standard Brownian motion processes independent of $W_t^e$.

The commercial property market-specific expected growth rate is also stochastic with the mean-reverting speed of $\lambda$, long term mean $\bar{\xi}$ and constant volatility $\sigma_\xi$.

$$d\xi_t = \lambda(\bar{\xi} - \xi_t)dt + \sigma_\xi dW_t^K \tag{13}$$

The economics of this mean-reversion specification is: people in the commercial property market usually make adjustments in supply and operating expenses in response to the observed return, which drives the expected return towards a long term mean.

We can see that at each point in time, property NOI is determined by four components: expected growth rate of the whole economy, commercial property market expected growth rate, economy-wide shock and commercial property market-specific shock.

The primitive processes in my model are economic growth and commercial property NOI growth processes rather than interest rate and property value processes, as in most existing structural models. Here, interest rate process is derived from the general equilibrium of the economy rather than exogenously given. Regarding NOI and property value, we know that commercial property value is largely determined by its NOI, so directly modelling NOI is potentially a better choice. In the following subsection, I'll come back to this when I discuss the so-called "double-trigger" of commercial mortgage default.

### 3.3 Commercial Mortgage Default Boundary

A typical commercial mortgage contract specifies the principal $M$, a fixed mortgage rate (coupon rate) $R$ and an amortization term $T$. The monthly principal and interest payment $C$ is determined implicitly by

$$M = \int_0^T C \exp(-Rt)dt \tag{14}$$

[11] I assume the existence of commercial properties and commercial mortgages do not affect equilibrium pricing given commercial properties and commercial mortgages only account for a tiny portion of the macro economy

Every month before maturity, the mortgage borrower is obligated to make the monthly payment. Usually commercial mortgages are balloon loans, with a maturity term $S < T$. At balloon date, the remaining principal is

$$M_S = \int_S^T C \exp(-Rt) dt \tag{15}$$

A commercial mortgage borrower usually default because he is unable or unwilling to make the monthly payment. A natural candidate of default boundary is the monthly payment amount. However, given high transaction cost of default, I follow Huang and Huang (2003) and Leland (2004) to specify a slightly different default boundary

$$K_B = \phi C \tag{16}$$

where $C$ is the monthly mortgage payment defined above. This boundary condition specifies that a commercial mortgage will be defaulted if the NOI of the underlying property falls under a certain proportion of the monthly mortgage payment. The parameter $\phi \leq 1$ is a control of transaction cost of mortgage default.

In the commercial mortgage credit risk literature, an important concern is the "double trigger" of default, which means the insolvency condition $K_B = C$ and the net worth condition $V_B = H$ are both important for default (see Riddiough 1991, Goldberg and Capone 2002 and Tu and Eppli 2002). Here, I only consider the insolvency condition. This is partly for the tractability of the model. However, there are other good reasons to focus on the insolvency condition. First, commercial property value is calculated as its cash flow divided by the market capitalization rate and is largely determined by its NOI. Therefore, the insolvency condition and the net worth condition are closely related. Second, from a practical point of view, the tangible monthly cash flow is well reflected on the mortgage holders accounting book, and thus it is easy to believe that mortgage holders make their default decisions based on observable cash flow conditions. This is consistent with the fact that insolvency condition expressed as debt-service coverage ratio (DSCR) is found to be the most relevant for commercial mortgage default event according to Moody's.

The default boundary here is assumed to be exogenously given. A more elegant treatment should specify an endogenously determined default boundary as in Leland (1994), Leland and Toft (1996) and Duffie and Lando (2001). However, in the above mentioned endogenous default boundary studies, a constant default boundary proportional to the debt coupon payment is usually derived based on assumptions such as constant interest rate, debt roll over and constant risk premium (see, for example, Duffie and Lando 2001, p.638). Therefore, I stick to the parsimonious specification of an exogenously given constant default boundary.

### 3.4 First-Passage Time Density

We want to know default probabilities of the commercial mortgage at each point in time before its maturity. Given the above set up, it is straightforward to find out the default time density, which is the first-passage time density of the stochastic property cash flow $K_t$ falling under the constant default boundary $K_B$. The following proposition gives the first-passage time density under real measure.

**Proposition 1:** The first-passage time density $q(\tau)$ is defined implicitly in the following integral equation

$$N\left(\frac{-\ln X_0 - L(t)}{\Omega(t)}\right) = \int_0^t q(\tau) N\left(\frac{L(\tau) - L(t)}{\Omega(t) - \Omega(\tau)}\right) d\tau \tag{17}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution function, and

$$\begin{aligned}
X_t &= \frac{K_t}{K_B} \\
L(t) &= \left(\beta\mu_0 + \xi_0 + \beta\overline{\mu} + \overline{\xi} - \frac{\sigma_K^2}{2} - \frac{\sigma_K\rho\beta\sigma_\mu}{\kappa} - \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\right) t \\
&\quad -\beta\overline{\mu}\left(e^{-\kappa t} - 1\right) - \overline{\xi}\left(e^{-\lambda t} - 1\right) + \frac{\sigma_K\rho\beta\sigma_\mu}{\kappa}\left(e^{-\kappa t} - 1\right) + \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(e^{-\lambda t} - 1\right) \\
\Omega(t) &= \frac{\beta^2\sigma_\mu^2}{\kappa^2}\left(t - 2\frac{1-e^{-\kappa t}}{\kappa} + \frac{1-e^{-2\kappa t}}{2\kappa}\right) + \frac{2\sigma_K\rho\beta\sigma_\mu}{\kappa}\left(t - \frac{1-e^{-\kappa t}}{\kappa}\right) \\
&\quad + \frac{\sigma_\xi^2}{\lambda^2}\left(t - 2\frac{1-e^{-\lambda t}}{\lambda} + \frac{1-e^{-2\lambda t}}{2\lambda}\right) + \frac{2\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(t - \frac{1-e^{-\lambda t}}{\lambda}\right) + 2\sigma_K^2 t
\end{aligned}$$

*Proof:* See appendix A.

Dividing the period from time zero to time T into $n$ equal sub-periods and discretizing the above integral equation give the following system of linear equations, which can be solved recursively to find out the default time density.

$$\begin{aligned}
N(a_i) &= \sum_{j=1}^{i} q_i N(b_{i,j}), \quad i = 1, 2, \cdots, n \\
q_i &= q\left(i\frac{T}{n}\right) \cdot \frac{T}{n}
\end{aligned} \tag{18}$$

### 3.5 Default Hazard Rate (Default Intensity)

Default hazard rate is a key variable studied in the mortgage default prediction literature. It is the mean arrival rate of default during $[t, s]$ conditioning on the mortgage not defaulted at time $t$ [12]. In real world, we observe default hazard rate based on pools of

[12] In discrete time, it is just the conditional default probability.

mortgages. We want to solve the relationship between default hazard rate and the risk factors in our model, so that we can use the real world data to estimate parameters of the model and learn the dynamics of risk factors and hazard rates. The following proposition gives the closed-form solution of default hazard rate in my model.

**Proposition 2:** The default hazard rate under real measure is

$$h_t = \frac{Pr(t \leq T < t+1; \mu_t, \xi_t)}{1 - Pr(T < t; \mu_t, \xi_t)} \tag{19}$$

where

$$\begin{aligned} Pr(t \leq T < t+1; \mu_t, \xi_t) \quad &= \int_1^{+\infty} \int_0^1 \frac{|\ln x_t + a_t|}{\sqrt{2\pi s^3}\sigma_K} \exp\left\{-\frac{(\ln x_t + a_t)^2}{2s\sigma_K^2}\right\} ds \\ &\quad \frac{1}{\sqrt{2\pi t}\sigma_K} \exp\left\{-\frac{(\ln x_t - \ln x_0 - \sum_{i=1}^{t} a_i)^2}{2t\sigma_K^2}\right\} dx_t \end{aligned}$$

$$Pr(T < t; \mu_t, \xi_t) = Pr(T < 1) + Pr(1 \leq T < 2) + \cdots + Pr(t-1 \leq T < t)$$

and

$$a_t = \beta\mu_t + \xi_t - \frac{\sigma_K^2}{2} + \frac{\sigma_K \rho\beta\sigma_\mu}{\kappa}\left(e^{-\kappa t} - 1\right) + \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(e^{-\lambda t} - 1\right)$$

*Proof:* see appendix B.

To make the relationship between hazard rate and the two state variables easy to see, we can write the hazard function as

$$h_t = f(\mu_t, \xi_t; \sigma_K, \sigma_\mu, \sigma_\xi, \rho, \beta, \kappa, \lambda, \overline{\mu}, \overline{\xi}, \phi) \tag{20}$$

where $f$ is a nonlinear function that maps the two state variables $\mu_t$ and $\xi_t$ into hazard rate $h_t$.

For pricing purposes, we can solve default probabilities under risk neutral measure in my model. The only thing we need to do is to transform the expressions of primitive processes from real measure into risk neutral measure using change of measure based on the result of market price of risk in equation (9). Then we can apply the same approach as above. Here I only give the hazard rate solutions in real measure because in the following empirical estimations, I will focus on default probabilities observed from real world commercial mortgage default experience.

# 4 Empirical Methodology

The objectives are to learn the evolutions of default risk over time based on observed commercial mortgage defaults, and to extract information about the latent systematic factors driving the default risk dynamics. Solutions of the structural model in the previous section can be put into a state space form and we can use real world data to estimate the model. To help identify the model, I use risk-free bond yield as another measurement variable in addition to default hazard rate.

## 4.1 The State Space Model

At time $t$ we observe commercial mortgage default hazard rate $h_t$ and risk-free bond yield $R_t$. These two observable variables are functions of the two latent state variables $\mu_t$ and $\xi_t$. Denote $Z_t = (R_t, h_t)'$ and $S_t = (\mu_t, \xi_t)'$, we can express the above structural model in a state-space form. Suppressing the dependence of the model on the parameters to be estimated, the measurement and transition equations are

$$Z_t = z(S_t) + \epsilon_t; \qquad E_{t-1}(\epsilon_t {\epsilon_t}') = \Sigma \tag{21}$$

$$S_t = \alpha + TS_{t-1} + \eta_t; \qquad E_{t-1}(\eta_t {\eta_t}') = \Phi \tag{22}$$

The transition equation (22) depicts the dynamics of the two state variables given in equations (2) and (13). They are simple $AR(1)$ processes. Notice $S_t$ is not directly observable. The non-linear function $z(S_t)$ in the measurement equation (21) maps the two latent state variables into the observable bond yield and mortgage hazard rate. Therefore, we can infer the dynamics of the latent variables from the time series of the measurement variables. The innovations in the transition equation are normally distributed based on our model set up in equations (2) and (13). I further assume the innovations in the measurement equation are also normally distributed, reflecting the random observation errors of these two variables.

The model is estimated with extended Kalman Filter as discussed in below. To apply the extended Kalman filter, I use the first-order Taylor expansion to linearize the measurement equation (21)

$$z(S_t) \approx z(S_t^*) - L_t S_t^* + L_t S_t \tag{23}$$

where

$$L_t = \left.\frac{\partial z(S_t)}{\partial S_t}\right|_{S_t = S_t^*}$$

## 4.2 The Cox Proportional Hazard Model

In real world, commercial mortgage loans are heterogeneous, e.g. they are in different region, backed by different properties, and have different mortgage terms. Existing studies have found loan characteristics to be important determinants of commercial mortgage default. Therefore, I use the Cox proportional hazard model to control for covariates of individual mortgage loans and estimate the hazard rate time series of a representative commercial mortgage.

The Cox proportional hazard model is widely accepted in the literature as an effective tool to study mortgage prepayment and default (See, for example, Green and Shoven 1987, Deng, Quigley and Van Order 2000). In the Cox model, the life of a mortgage loan (duration $T$) is the random variable which is usually assumed to follow a certain distribution. In each month, the mortgage loan has a risk (hazard) of ending its life.

The Cox model starts with the specification of the hazard function

$$h_i(t,T) = h_0(T)\exp(X_{i,t}\beta) \tag{24}$$

where $t$ is calendar time and $T$ is duration. $h_0(T)$ is the baseline hazard function shared by all loans. It represents the default rate of a typical loan during the interval $[T, T+1]$ conditioning on not defaulted at duration $T$. $X_{i,t}$ are covariates to control heterogeneity of individual mortgage loans. They shift the baseline hazard up or down depending on their values. Covariates can be both time constant and time-varying.

In each month, the hazard rate of a mortgage loan is determined by economic fundamentals, loan specific characteristics and seasoning. Therefore, we can further separate time varying covariates from time constant covariates, and write the hazard function as

$$h_i(t,T) = h_0(T)\exp(D_{i,t}\alpha + Z_i\beta) \tag{25}$$

where $Z$ is a vector representing time constant idiosyncratic risk factors such as property type, region and maturity term. $D_{i,t} = (d_{i,1}, d_{i,2}, \ldots, d_{i,t})$ is a series of time varying dummies and thus $\beta$ captures systematic risk, which evolves over time. $h_0(T)$ shared by all loans is a function of duration $T$, which represents the seasoning effect. Following Schwartz and Torous (1989), I assume the conditional distribution of the duration time $T$ (on the covariates) follows a log-logistic distribution and thus the baseline hazard function is[13]

$$h_0(T) = \frac{\nu\omega(\nu T)^{\omega-1}}{1+(\nu T)^{\omega}} \tag{26}$$

### 4.3 Estimation Methods

I estimate the Cox proportional hazard model using the maximum likelihood estimation method presented in Kalbfleisch and Prentice (1983/2002) and implemented by Deng, Quigley and Van Order (2000) [14].

[13] Although in the corporate credit risk literature some researchers argue that the observed seasoning effect might be a manifestation of the business cycle effect (see, for example Blume, Keim and Patel (1991) and Jonsson and Fridson (1996), there is general consensus that the seasoning effect prevails in mortgages.

[14] Deng, Quigley and Van Order (2000) have competing risks (prepayment and default) in their model and they estimate a flexible baseline using a semi-parametric method.

The MLE involves the survival function, which is

$$V_i(t,T) = \exp\left\{-\int_0^T h_i(t,s)ds\right\} \tag{27}$$

The likelihood function is

$$L = \prod_{l=0}^{T}\{\prod_{i\in D_i}(V_i(t,T+1) - V_i(t,T))\prod_{i\in C_i}V_i(t,T+1)\} \tag{28}$$

with $D_i$ be the set of labels associated with individuals failing at $t_l$, and $C_i$ be the set of labels associated with individuals censored in $[t_l, t_{l+1})$. The asymptotic covariance of the MLE estimates is calculated with the information matrix using the BHHH method (Greene 2000, p.132).

The hazard model estimation gives the time series of default hazard rate of a representative commercial mortgage $\widehat{H_t} = (\widehat{h_1}, \widehat{h_2}, \ldots, \widehat{h_t})$, which is the measurement variable $h_t$ in the state space model.

The state space model is estimated using the extended Kalman filter. The nonlinear function $z(S_t)$ is linearized with equation (23) at the 1-month ahead forecast of $S_t$ and $\Phi$ is evaluated at the contemporaneous prediction of $S_t$. After linearization, the standard Kalman filter recursion is used.

The Kalman filter requires initialization. Since we don't know the exact distribution of the initial state vector $S_1$, I apply a diffuse initialization, which assumes arbitrary means and infinite variances of the initial state vector.

Parameters of the state space model are estimated using maximum likelihood estimation method. The log-likelihood function with diffuse initialization is

$$LogL_d(Z) = -\frac{np}{2}\log 2\pi - \frac{1}{2}\log|F_{\infty,t}| - \frac{1}{2}\sum_{t=2}^{n}(\log|F_t| + v_t^{'}F_t^{-1}v_t) \tag{29}$$

where $n$ is the number of time series observations, $p$ is the measurement variable dimension, which is 2 in my model. $F_{\infty,t} = C_t P_{\infty,t} C_t^{'}$. $F_t$ is the prediction variance $Var(Z_t|\mathcal{F}_{t-1})$ with $\mathcal{F}_{t-1}$ denoting information up to $t-1$, and $v_t$ is the prediction error $Z_t - C_t S_t$. $P_{\infty,t} = T_t T_t^{'}$. Again, the asymptotic covariance of the MLE estimates is calculated using the BHHH method.

# 5 Data and Results

I access a large commercial mortgage loan performance database maintained by a mortgage data provider. Information are collected for nearly 60,000 commercial mortgage loans, which all underly commercial mortgage-backed securities (CMBS). The data collecting point is June 31, 2003. After excluding adjustable rate mortgages (ARMs) and mortgages in Canada, the final sample contains 49,389 fixed rate commercial mortgage loans. These loans are originated between 1992 and 2003, and are from 355 MSAs across 54 US states and territories. The data contains detailed information on loan characteristics and performance, such as origination date, original balance, original loan-to-value ratio (LTV), maturity date, amortization term, coupon rate, lender, property type, geography, 60-day and 90-day delinquency dates, pay down date, losses, etc..

In order to utilize the simple Affine term structure results in equation (11), we need yields of risk-free pure discount bond. I use the Fama-Bliss 5-year pure discount bond yields from CRSP, and apply the seasonally adjusted CPI to make the inflation indexed yields.

Table 1 shows the origination year distribution of the commercial mortgage sample with respect to both number of loans and original balance. The year 1998 sees a large number of loan origination. Loans originated after 2000 have substantially higher average loan amount than those originated in 1992 through 1996, even taking into consideration of inflation.

I identify 1,877 defaults, which is 3.80% of the whole sample [15]. Table 2 shows distribution of the sample with respect to different loan characteristics. I also calculate the proportion of loans defaulted in each group, as shown in the third column of the table. The most populated property type in my sample is multifamily, followed by retail and office. Hotel and healthcare property loans have substantially higher default rates. The loans in my sample are from 10 US regions, with those in the west coast to have lower default rates. Most loans have amortization terms between 20 and 30 years. Less than 6% of loans have very short (less than 10 years) or very long (over 30 years) amortization, and those loans have lower default rates. Different from residential mortgages, most commercial mortgages are balloon loans. Nearly 75% of loans in my sample mature in 5 to 10 years. Average LTV of these loans are also lower than residential mortgages. The majority of loans have LTV in the 60-80% range. Loans with very low LTV (lower than 40%) have significantly lower default rates. The average original balance is $6.14 million, and the top 5 percent of the sample have balances over $18 million. The simple tabulation in table 2 tends to suggest substantial variations in default rates with respect to loan characteristics.

Table 3 presents the maximum likelihood estimation results for the time constant covariates of the Cox proportional hazard model. Most of the results are conforming to expectation. For example, loans with LTV of over 80 percent have higher default probabilities than the majority of the sample, which have LTV between 60 and 80 percent. Hotel and

[15] Also excluded from the sample are 164 balloon default loans, since balloon default is not modelled in my model. Reasons for balloon defaults could be very different from those of term defaults, e.g. some borrowers default at maturity date because of the inability to refinance the current loan at the balloon date given a substantially higher interest rate environment than that at origination.

healthcare loans are more risky than the reference group - retail loans. Loans in Southern California (Western/Pacific) have the lowest risk across regions, while those in the South have higher risk comparing to the reference group - Northeast/Mid-Atlantic loans. The maturity term is negatively related with default risk. The negative relationship applies to amortization. These two relationships are consistent with the notion that properties with lower risk usually get more favorable loan terms. The only surprise in table 3 comes from multifamily, which is shown to have higher risk than retail properties. This is possibly because of the over-built of multifamily during the study period due to declining interest rate.

Figure 1 plots the estimated seasoning effect, which shows a humped shape. Default hazard rate increases rapidly in the first 3 (duration) years, reaches its maximum in four and a half years, and then declines gradually. This is generally consistent with findings of Esaki (2002), who find commercial mortgage loan default rate to increase with duration, reach the highest level in year 4, and then have small changes until year 7. Interestingly, the decline of commercial mortgage hazard rate after the peak year is much slower than that of residential mortgage. The magnitude of commercial mortgage hazard rate is apparently much higher than that of residential mortgage. The figure shows that for mortgages in a fair year, the peak default hazard rate (with respect to seasoning) is about 2.4%, while the 100% SDA has a peak hazard rate of only 0.6%. The 5 year and 7 year cumulative default probabilities are 2.35% and 5.06% based on these estimates.

As mentioned earlier, the focus of this paper is really on the time series dynamics of commercial mortgage default risk. Figure 2 shows the estimated hazard rate time series of a representative mortgage. At least four points can be taken from these results: first, there are large variations in default risk over time. The annual default hazard rate of a five-year seasoned commercial mortgage could be as low as 0.39% (16.11% of sample average) in good years, and could be as high as 5.78% (238.20% of sample average) in bad years. Second, the changes in default hazard rate is generally consistent with changes in macroeconomic conditions, e.g. default risk increase substantially during 2001 and 2002, when the economy enters a recession. Third, the observed high default hazard rate during 1997-1998 when the economy is strong points to the importance of the market-specific factor. In fact, large scales of foreign investment enters the US commercial real estate market in the 1990s, and the Asian financial crisis and the Russian financial crisis cause a lot of problems in those properties. Last, default hazard rate tends to have a cyclical movement, going down in 1994-1996, rising up in 1997-1998, and going down again in 1999-2000 and rising up again in 2001-2002.

Figure 3 plots the 5-year risk-free pure discount bond yields. The bond yields tend to move oppositely with the default hazard rates shown in figure 2. Notice that in my model low bond yield implies lower expected economic growth rate. Therefore, this is a sensible pattern.

Table 4 reports the core results of the paper. For identification purposes, I do not estimate all the free parameters in my model. Instead, I choose to pre-specify some of the parameters, e.g. the volatilities of economic growth and property income growth $\sigma_e$ and $\sigma_K$ are fixed at 8%, the risk aversion parameter $\gamma$ is chosen to be 4 and the subjective discount factor $\delta$ is 4%. These values are conforming to what we observe or what people use in the literature. The estimates show the existence of two significant mean-reverting unobservable state variables underlying commercial mortgage default. The expected economic growth has a long term mean of 1.70%, volatility of 0.63% and annual mean-reversion speed of 0.46. The commercial property market-specific factor has a long term growth rate of 1.20% and mean-reversion speed of 0.596. The volatility of this property market-specific factor is 0.6%. As shown in the table, dynamics of default hazard rate can be largely explained by the dynamics of these two mean-reverting latent variables. The root mean square error (RMSE) is less than 15% of the sample mean, which means changes of these two factors explain over 85 percent of the variations in default hazard rate.

Based on the model estimates, the long term mean of spot interest rate is 4.4%, and the long term mean of annual default hazard rate is 0.06% for a newly originated loan and 2.58% for a five-year seasoned loan. Table 5 also shows default hazard rates of a representative mortgage with different seasoning under a favorable environment and under a bad environment.

Simulations shown in figure 4 tells that commercial mortgage default risk is more sensitive when the economy is in downturn than at a time when the economic growth is high. There is also a negative relationship between default risk and interest rate, conforming to the existing literature (Duffee 1998 and Longstaff and Schwartz 1995). This provides hedging implications for portfolio credit risk.

The final point to take from the empirical estimates is that the insolvency threshold for default is 89% of the monthly mortgage payment, assuming an underwriting debt-service-coverage ratio (DSCR) of 1.3. This is calculated from the relationship $x_0 = \frac{K_0}{\phi C} = \frac{DSCR}{\phi}$. Since the estimated $x_0$ is 1.46, $\phi$ is 0.89. This result is consistent with the observation that mortgage borrowers do not default ruthlessly possibly because of substantial transaction cost of default.

## 6 Conclusions and Discussions

There are increasing concerns of the time series dynamics of credit risk among both academic researchers and industrial practitioners. This is because credit risk not only varies across firms or borrowers, but also changes over time depending on some macroeconomic and systematic risk factors. Understanding the time series dynamics of credit risk and their underlying risk factors is important to portfolio risk management, hedging and pricing.

The mortgage credit risk literature has extensively examined the cross sectional differences in default risk, but left the time series properties of default risk unexplored. I intend to fill this gap by studying the time series dynamics of commercial mortgage credit risk and the unobservable systematic risk factors underlying those dynamics.

I present a first-passage structural model for commercial mortgage credit risk and solve the default hazard rate under real measure. The model is then estimated with commercial mortgage default data using extended Kalman filter. The estimates show the existence of two significant mean-reverting latent factors underlying commercial mortgage default. The expected economic growth is one factor. In addition, there is a significant commercial property market-specific factor. The variations in default hazard rate are well explained by the dynamics of the two mean-reverting latent factors. Empirical estimates also confirm the notion that there is substantial transaction cost of default.

From a methodological perspective, I bridge the gap between theoretical credit risk models and empirical default estimations by providing an integrated credit risk model.

Future research can go several directions. First, more measurement variables can be brought in to identify all the free parameters in the model, e.g. incorporating the dynamics of the whole term structure of interest rate will provide additional degrees of freedom. Second, more elements can be incorporated into the current theoretical framework, e.g. an exogenously given inflation process as modelled in Wachter (2006) is desirable, and mean-reverting asset risk premium (Huang and Huang, 2003) might also better approximate real world situations. Third, pricing is definitely an interesting topic. Given the current structure of the model, we can extend the study into commercial mortgage pricing once data is available. In addition, an important application of my model will be in the residential mortgage market. In fact, with much longer data series in residential mortgage market, we can perform empirical tests of credit cycles. Last, we may empirically improve the current state-space model by allowing non-Gaussian disturbances. Importance sampling can be used to estimate such models (Durbin and Koopman 2001 and Koopman, Lucas and Daniel 2005).

## References


Allen, L. and A. Saunders, 2003, A Survey of Cyclical Effects in Credit Risk Measurement Models, BIS Working Paper No. 126.

Altman, E. I, 1968, Financial Ratios, Discriminant Analysis and the Prediction of the Corporate Bankruptcy, *Journal of Finance* 23(4), 589-609.

Altman, E. I, 1990, *Corporate Financial Distress*, Wiley, New York.

Ambrose, B. W. and A. B. Sanders, 2003, Commercial Mortgage Backed Securities: Prepayment and Default, *Journal of Real Estate Finance and Economics* 26(2/3), 179-196.

An, X., J. C. Clapp and Y. Deng, 2005, Omitted Mobility Characteristics and Property Market Dynamics: Application to Mortgage Termination, USC Lusk Center for Real Estate Working Paper, No. 2005-1007.

Bai, J. and S. Ng, 2004, Evaluating Latent and Observed Factors in Macroeconomics and Finance, EconWPA Working Paper No. 0408007.

Black, F. and J. C. Cox, 1976, Valuing Corporate Securities: Some Effects of Bond Indenture provisions, *Journal of Finance* 31(2), 351-367.

Blume, M. and D. Keim, 1991, Realized Returns and Defaults on Low-Grade Bonds: The Cohort of 1977 and 1978, *Financial Analysts Journal* 47(2), 63-72.

Blume, M., D. Keim and S. Patel, 1991, Returns and Volatility of Low-Grade Bonds: 1977-1989, *Journal of Finance* 46(1): 49-74.

Buonocore, A., A. G. Nobile and L. M. Ricciardi, 1987, A New Integral Equation for the Evaluation of First-passage-time Probability Densities, *Advances in Applied Probability* 19, 784-800.

Campbell, T. and J. K. Dietrich, 1983, The Determinants of Default on Conventional Residential Mortgages, *Journal of Finance* 48(5), 1569-1581.

Campbell J. Y., J. Hilscher and J. Szilagyi. In Search of Distress Risk. Harvard Institute of Economic Research Discussion Paper No. 2081.

Capozza, D. R., D. Kazarian and T. A. Thomson, 1997, Mortgage Default in Local Markets, *Real Estate Economics* 25(4), 631-655.

Carey, M., 1998, Credit Risk in Private Debt Portfolios, *Journal of Finance* 53(4), 1363-1387.

Childs, P. D., S. H. Ott and T. J. Riddiough, 1996, The Pricing of Multiclass Commercial Mortgage-Backed Securities, *Journal of Financial and Quantitative Analysis* 31(4), 581-603.

Clapp, J. M., Y. Deng and X. An, 2006, Unobserved Heterogeneity in Models of Competing Mortgage Termination Risks, *Real Estate Economics* 34(2), 243-273.

Clapp, J. C., G. M. Goldberg, J. P. Harding and M. LaCour-Little, 2001, Movers and Shuckers: Interdependent Prepayment Decisions, *Real Estate Economics* 29(3), 411-450.

Collin-Dufresne, P. and R. Goldstein, 2001, Do Credit Spread Reflect Stationary Leverage Ratios? *Journal of Finance* 56(5), 1929-1958.

Cunningham, C. R. and P.H. Hendershott, 1984, Pricing FHA Mortgage Default Insurance, *Housing Finance Review* 3(4), 373-392.

Deng, Y. and J. M. Quigley, 2002, Woodhead Behavior and the Pricing of Residential Mortgages, USC Lusk Center for Real Estate Working Paper No. 2001-1005.

Deng, Y., J. M. Quigley and R. Van Order, 2000, Mortgage Terminations, Heterogeneity and the Exercise of Mortgage Options, *Econometrica* 68(2), 275-307.

Deng, Y., A. D. Pavlov and L. Yang, 2005, Spatial Heterogeneity in Mortgage Terminations by Refinance, Sale and Default *Real Estate Economics* 33(4), 739-764.

Downing, C., R. Stanton and N. Wallace, 2005, An Empirical Test of a Two-Factor Mortgage Valuation Model: How Much Do House Prices Matter? *Real Estate Economics* 33(4), 681-710.

Duffee, G., 1998, The Relationship between Treasury Yields and Corporate Bond Yield Spreads, *Journal of Finance* 53(6), 2225-2241.

Duffee, G., 1999, Estimating the Price of Default Risk, *Review of Financial Studies* 12(1), 197-226.

Duffie, D. and D. Lando, 2001, Term Structures of Credit Spreads with Incomplete Accounting Information, *Econometrica* 69(3), 633-664.

Duffie, D. and K. J. Singleton, 1999, Modeling Term Structures of Defaultable Bonds, *Review of Financial Studies* 12(4), 687-720.

Duffie, D. and K. J. Singleton, 2003, *Credit Risk: Pricing, Management, and Measurement* Princeton University Press, New Jersey.

Duffie, D., K. Wang and L. Saita, 2006, Multi-period Corporate Default Prediction with Stochastic Covariates, NBER Working Paper No. 11962.

Durbin, J. and S. J. Koopman, 2001, *Time Series Analysis by State Space Methods* Oxford University Press, New York.

Esaki, H., 2002, Commercial Mortgage Defaults: 1972-2000, *Real Estate Finance* 2002(Winter), 43-52.

Foster, C. and R. Van Order, 1984, An Option-Based Model of Mortgage Default, *Housing Finance Review* 3(4), 351-372.

Gabriel, S. A., J. P. Rothberg and F. E. Nothaft, 1989, On the Determinants of Yield Spreads Between Mortgage Pass-Through and Treasury Securities, *Journal of Real Estate Finance and Economics* 2, 301-15.

Giliberto, S. M. and D. C. Ling, 1992, An Empirical Investigation of the Contingent-Claims Approach to Pricing Residential Mortgage Debt, *Journal of American Real Estate and Urban Economics Association* 20(3), 393-426.

Goldberg, L. and C. A. Capone Jr., 2002, A Dynamic Double-Trigger Model of Multifamily Mortgage Default, *Real Estate Economics* 30(1), 85-113.

Goldstein, R. and F. Zapatero, 1996, General Equilibrium with Constant Relative Risk Aversion and Vasicek Interest Rates,*Mathematical Finance* 6(3), 331-340.

Gordy, M. B., 2003, A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules, *Journal of Financial Intermediation* 12(3), 199-232.

Greene, W. H., 2000, *Econometric Analysis* (4th edition) Prentice Hall, New Jersey.

Grimmett, G. R. and D. R. Stirzaker, 2001, *Probability and Random Processes*, Oxford University Press.

Huang, J. and M. Huang, 2003, How Much of the Corporate-Treasury Yield Spread is Due to Credit Risk? Stanford University Working Paper.

Hose, S. and K. Vogl, 2005, Predicting the Credit Cycle with an Autoregressive Model, Working Paper.

Jarrow, R., D. Lando and S. Turnbull, 1997, A Markov Model For the Term Structure of Credit Risk Spreads, *Review of Financial Studies* 10(2), 481-523.

Jarrow, R. and S. Turnbull, 1995, Pricing Derivatives on Financial Securities Subject to Credit Risk, *Journal of Finance* 50(1), 53-86.

Jarrow, R. and S. Turnbull, 2000, The Interaction of Market and Credit Risk, *Journal of Banking and Finance* 24, 271-299.

Kalbfleisch, J. D. and R. L. Prentice, 1983/2002, *The Statistical Analysis of Failure Time Data*(2nd edition) Wiley, New Jersey.

Kau, J. B., D. C. Keenan, W. J. Muller III and J. F. Epperson, 1987, The Valuation and Securitization of Commercial and Multifamily Mortgages, *Journal of Banking and Finance* 11, 525-546.

Kau, J. B., D.C. Keenan, and A. A. Smurov, 2006, Reduced Form Mortgage Pricing as an Alternative to Option-Pricing Models, *Journal of Real Estate Finance and Economics* 33(3), 183-196.

Koopman S. J., A. Lucas, 2005, Business and Default Cycles for Credit Risk, *Journal of Applied Econometrics* 20, 311-323.

Koopman, S. J., A. Lucas and P. Klaassen, 2005, Empirical Credit Cycles and Capital Buffer Formation. *Journal of Banking and Finance* 29(12), 3159-3179.

Lando, D., 1998, Cox Processes and Credit-Risky Securities, *Review of Derivative Research* 2, 99-120.

Lee, S. H. and J. L. Urrutia, 1996, Analysis and Prediction of Insolvency in the Property-Liability Insurance Industry: A Comparison of Logit and Hazard Models, *Journal of Risk and Insurance* 63, 121-130.

Leland, H., 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance* 49(4), 1213-1252.

Leland, H., Predictions of Default Probabilities in Structural Models of Debt, University of California, Berkeley Working Paper.

Leland, H. and K. B. Toft, 1996, Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads, *Journal of Finance* 51(3), 987-1019.

Lennox, C, 1999, Identifying Failing Companies: A Re-Evaluation of the Logit, Probit and DA Approaches, *Journal of Economics and Business* 51, 347-364.

Longstaff, F. A. and E. S. Schwartz, 1995, A Simple Approach to Valuing Risky and Floating Rate Debt, *Journal of Finance* 50(3), 789-819.

Merton, R. C., 1974, On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance* 29(2), 449-470.

Mandan, D. B., and H. Unal, 1996, Pricing the Risks of Default, Paper Presented at the Wharton Financial Institutions Center's Conference on Risk Management in Banking, October 13-15, 1996.

Mandan, D. B., and H. Unal, 2000, A Two-factor Hazard Rate Model for Pricing Risky Debt and the Term Structure of Credit Spreads, *Journal of Financial and Quantitative Analysis* 35(1), 43-65.

Maxam, C. and J. Fisher, 2001, Pricing Commercial Mortgage Backed Securities, *Journal of Property Investment and Finance* 19(6), 498-518.

McDonald, C. G. and L. M. Van de Gucht, 1999, High-Yield Bond Default and Call Risks, *Review of Economics and Statistics* 81(3), 409-419.

Merton, R. C, 1974, On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, *Journal of Finance* 29(2), 449-470.

Ohlson, J., 1980, Financial Ratios and the Probabilistic Prediction of Bankruptcy, *Journal of Accounting Research* 19, 109-131.

Pesaran, M. H., T. Schuermann, B. Treutler and S. M. Weiner, 2005, Macroeconomics and Credit Risk: A Global Perspective, *Journal of Money, Credit and Banking*

Quigley, J. M. and R. Van Order, 1995, Explicit Tests of Contingent Claims Models of Mortgage Default, *The Journal of Real Estate Finance and Economics* 1(2), 99-117.

Riddiough, T. J., 1991, Equilibrium Mortgage Default Pricing with Non-Optimal Borrower Behavior, University of Wisconsin Ph.D. Dissertation, Madison, WI.

Riddiough, T. J., 2004, Commercial Mortgage-backed Securities: An Exploration into Agency, Innovation, Information, and Learning in Financial Markets, Mimeo, University of Wisconsin, Madison, WI.

Riddiough, T. J. and H. E. Thompson, 1993, Commercial Mortgage Default Pricing with Unobservable Borrower Default Costs, *Journal of the American Real Estate and Urban Economics Association* 21(3), 265-291.

Schwartz, E. and W. N. Torous, 1989, Prepayment and the Valuation of Mortgage-backed Securities, *Journal of Finance* 44(2), 375-392.

Schwartz, E. and W. N. Torous, 1992, Prepayment, Default, and the Valuation of Mortgage Pass-through Securities, *Journal of Business* 65(2), 221-239.

Shumway, T., 2001, Forecasting Bankruptcy More Accurately: A Simple Hazard Model, *Journal of Business* 74(1), 101-124.

Stambaugh, R., 1988, The Information in Forward Rates: Implications for Models of the Term Structure, *Journal of Financial Economics* 21, 41-70.

Tang, D. Y. and H. Yan, 2005, Macroeconomic conditions, firm characteristics, and credit spreads, *Journal of Financial Services Research* 29, 177-210.

Titman, S., and W. N. Torous, 1989, Valuing Commercial Mortgages: An Empirical Investigation of the Contingent-Claims Approach to Pricing Risky Debt, *Journal of Finance*, 44(2), 345-373.

Tu, C. and M. J. Eppli, 2002, Pricing Credit Risk in Commercial Mortgages, Working Paper.

Van Order, R. and P. Zorn, 2001, Performance of Low-income and Minority Mortgages, Harvard University Joint Center for Housing Studies Working Paper No. LIHO-01.10.

Vandell, K. D, 1992, Predicting Commercial Mortgage Foreclosure Experience, *Journal of the American Real Estate and Urban Economics Association* 20(1), 55-88.

Vandell, K. D., W. Barnes, D. Hartzell, D. Kraft and W. Wendt, 1993, Commercial Mortgage Defaults: Proportional Hazards Estimation Using Individual Loan Histories, *Journal of the American Real Estate and Urban Economics Association* 21(4), 451-480.

Vandell, K. D. and T. Thibodeau, 1985, Estimation of Mortgage Defaults Using Disaggregate Loan History Data, *Journal of the American Real Estate and Urban Economics Association* 13(3), 292-316.

von Furstenberg, G., 1969, Default Risk on FHA-Insured Home Mortgage as a Function of the Term of Financing: A Quantitative Analysis, *Journal of Finance* 24(2), 459-77.

Wachter, J. A., 2006, A Consumption-based Model of the Term Structure of Interest Rates, *Journal of Financial Economics* 79, 365-399.

Wilson, T. C., 1997, Portfolio Credit Risk (I and II), *Risk*, 10(September and October), 111-117, 56-61.

Wilson, T. C., 1998, Portfolio Credit Risk, FRBNY Economic Policy Review 1998(October), 71-82.

# A Proof of Proposition 1

We have the set up in equations (12), (13) and (2) as follows

$$\frac{dK_t}{K_t} = (\beta\mu_t + \xi_t)dt + \sigma_K \rho dW_t^e + \sigma_K \sqrt{1-\rho^2} dW_t^K \tag{30}$$

$$d\mu_t = \kappa(\overline{\mu} - \mu_t)dt + \sigma_\mu dW_t^e \tag{31}$$

$$d\xi_t = \lambda(\overline{\xi} - \xi_t)dt + \sigma_\xi dW_t^K \tag{32}$$

Define $X_t = \frac{K_t}{K_B}$, then

$$\begin{aligned} \tfrac{dX_t}{X_t} &= (\beta\mu_t + \xi_t)dt + \sigma_K \rho dW_t^e + \sigma_K \sqrt{1-\rho^2} dW_t^K \qquad (33) \\ \ln X_t - \ln X_0 &= \int_0^t \left[\beta\mu_s + \xi_s - \frac{\sigma_K^2}{2} + \frac{\sigma_K \rho \beta \sigma_\mu}{\kappa}\left(e^{-\kappa s} - 1\right)\right. \\ &\left. + \frac{\sigma_K \sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(e^{-\lambda s} - 1\right)\right] ds + \int_0^t \left[\sigma_K \rho dW_s^e + \sigma_K \sqrt{1-\rho^2} dW_s^K\right] \end{aligned}$$

Rewrite equation (31) and (32) as

$$\begin{aligned} \mu_t &= \mu_0 + e^{-\kappa t}\int_0^t \kappa\overline{\mu} e^{\kappa s} ds + e^{-\kappa t}\int_0^t \sigma_\mu e^{\kappa s} dW_s^e \qquad (34) \\ \xi_t &= \xi_0 + e^{-\lambda t}\int_0^t \lambda\overline{\xi} e^{\lambda s} ds + e^{-\lambda t}\int_0^t \sigma_\xi e^{\lambda s} dW_s^K \end{aligned}$$

Substitute $\mu_s$ and $\xi_s$ in equation (33) by the above, and rearrange items, we have

$$\begin{aligned} \ln X_t - \ln X_0 &= \int_0^t \left[\beta\mu_0 + \xi_0 - \frac{\sigma_K^2}{2} - \frac{\sigma_K \rho\beta\sigma_\mu}{\kappa}(1 - e^{-\kappa s})\right. \qquad (35) \\ &\left. - \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}(1 - e^{-\lambda s})\right] ds \\ &+ \int_0^t \left[\beta e^{-\kappa s}\int_0^t \kappa\overline{\mu} e^{\kappa q} dq + e^{-\lambda s}\int_0^t \lambda\overline{\xi} e^{\lambda q} dq\right] ds \\ &+ \int_0^t \left[\beta e^{-\kappa s}\int_0^t \sigma_\mu e^{\kappa q} dW_q^e + e^{-\lambda s}\int_0^t \sigma_\xi e^{\lambda q} dW_q^K\right] ds \\ &+ \int_0^t \left[\sigma_K \rho dW_s^e + \sigma_K\sqrt{1-\rho^2} dW_s^K\right] \end{aligned}$$

Notice

$$\begin{aligned} \textstyle\int_0^t e^{-\kappa s}\int_0^t e^{\kappa q} dW_q^e ds &= \int_0^t dW_q^e \int_0^t e^{-\kappa(q-s)} ds \qquad (36) \\ &= \int_0^t dW_q^e e^{\kappa q}\frac{e^{-\kappa t} - e^{-\kappa q}}{\kappa} = \int_0^t \frac{1 - e^{\kappa(q-t)}}{\kappa} dW_q^e \end{aligned}$$

Hence, equation (35) can be further written as

$$\begin{aligned}\ln X_t - \ln X_0 &= \left(\beta\mu_0 + \xi_0 + \beta\overline{\mu} + \overline{\xi} - \frac{\sigma_K^2}{2} - \frac{\sigma_K\rho\beta\sigma_\mu}{\kappa} - \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\right)t \qquad (37)\\ &-\beta\overline{\mu}\left(e^{-\kappa t}-1\right) - \overline{\xi}\left(e^{-\lambda t}-1\right)\\ &+\frac{\sigma_K\rho\beta\sigma_\mu}{\kappa}\left(e^{-\kappa t}-1\right) + \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(e^{-\lambda t}-1\right)\\ &+\int_0^t\left[\frac{\beta\sigma_\mu}{\kappa}\left(1-e^{\kappa(q-t)}\right)dW_q^e + \frac{\sigma_\xi}{\lambda}\left(1-e^{\lambda(q-t)}\right)dW_q^K\right]\\ &+\int_0^t\left[\sigma_K\rho dW_s^e + \sigma_K\sqrt{1-\rho^2}dW_s^K\right]\\ &= L(t) + \int_0^t\left[\frac{\beta\sigma_\mu}{\kappa}\left(1-e^{\kappa(q-t)}\right)dW_q^e + \frac{\sigma_\xi}{\lambda}\left(1-e^{\lambda(q-t)}\right)dW_q^K\right]\\ &+\int_0^t\left[\sigma_K\rho dW_s^e + \sigma_K\sqrt{1-\rho^2}dW_s^K\right]\end{aligned}$$

Therefore,

$$E(\ln X_t) = \ln X_0 + L(t) \qquad (38)$$

and

$$\begin{aligned}Var(\ln X_t) &= \int_0^t\left[\frac{\beta\sigma_\mu}{\kappa}\left(1-e^{\kappa(q-t)}\right) + \sigma_K\rho\right]^2 dq \qquad (39)\\ &+\int_0^t\left[\frac{\sigma_\xi}{\lambda}\left(1-e^{\lambda(q-t)}\right) + \sigma_K\sqrt{1-\rho^2}\right]^2 dq + \sigma_K^2 t\\ &= \int_0^t\left[\frac{\beta^2\sigma_\mu^2}{\kappa^2}\left(1-e^{\kappa(q-t)}\right)^2 + \frac{2\sigma_K\rho\beta\sigma_\mu}{\kappa}\left(1-e^{\kappa(q-t)}\right)\right]dq\\ &+\int_0^t\left[\frac{\sigma_\xi^2}{\lambda^2}\left(1-e^{\lambda(q-t)}\right)^2 + \frac{2\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(1-e^{\lambda(q-t)}\right)\right]dq + 2\sigma_K^2 t\end{aligned}$$

which can be further written as

$$\begin{aligned}Var(\ln X_t) &= \frac{\beta^2\sigma_\mu^2}{\kappa^2}\left(t - 2\frac{1-e^{-\kappa t}}{\kappa} + \frac{1-e^{-2\kappa t}}{2\kappa}\right) \qquad (40)\\ &+\frac{2\sigma_K\rho\beta\sigma_\mu}{\kappa}\left(t - \frac{1-e^{-\kappa t}}{\kappa}\right) + \frac{\sigma_\xi^2}{\lambda^2}\left(t - 2\frac{1-e^{-\lambda t}}{\lambda} + \frac{1-e^{-2\lambda t}}{2\lambda}\right)\\ &+\frac{2\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(t - \frac{1-e^{-\lambda t}}{\lambda}\right) + 2\sigma_K^2 t = \Omega(t)\end{aligned}$$

So, $\ln X_t \sim N\left(\ln X_0 + L(t), \Omega(t)\right)$, and according to Buonocore, Nobile and Ricciardi (1987, 2.2a), the first-passage time density $q(\tau)$ is defined implicitly in the following integral equation [16]

$$N\left(\frac{-\ln X_0 - L(t)}{\Omega(t)}\right) = \int_0^t q(\tau) N\left(\frac{L(\tau) - L(t)}{\Omega(t) - \Omega(\tau)}\right) d\tau \tag{41}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution function.

Q.E.D.

# B Proof of Proposition 2

Define $X_t = \frac{K_t}{K_B}$, again we have

$$\begin{aligned}
\frac{dX_t}{X_t} &= (\beta\mu_t + \xi_t)dt + \sigma_K \rho dW_t^e + \sigma_K\sqrt{1-\rho^2}dW_t^K \quad (42)\\
\ln X_t - \ln X_0 &= \int_0^t \left[\beta\mu_s + \xi_s - \frac{\sigma_K^2}{2} + \frac{\sigma_K\rho\beta\sigma_\mu}{\kappa}\left(e^{-\kappa s} - 1\right)\right.\\
&\left. + \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(e^{-\lambda s} - 1\right)\right] ds + \int_0^t \left[\sigma_K\rho dW_s^e + \sigma_K\sqrt{1-\rho^2}dW_s^K\right]
\end{aligned}$$

Further defining

$$a_t = \beta\mu_t + \xi_t - \frac{\sigma_K^2}{2} + \frac{\sigma_K\rho\beta\sigma_\mu}{\kappa}\left(e^{-\kappa t} - 1\right) + \frac{\sigma_K\sqrt{1-\rho^2}\sigma_\xi}{\lambda}\left(e^{-\lambda t} - 1\right) \tag{43}$$

and orthogonizing the Brownian motions in the last integration term, we get

$$\ln X_t - \ln X_0 = \int_0^t a_s ds + \int_0^t \sigma_K dW_s \tag{44}$$

So,

$$\ln X_{t+1} - \ln X_t = a_t + \sigma_K W(1) \tag{45}$$

or

$$\frac{\ln X_{t+1} - \ln X_t - a_t}{\sigma_K} = W(1) \sim N(0,1)$$

Since $X_{t+1} = 1$ is the boundary condition, according to Grimmett and Stirzaker (2001, p. 526), the first-passage time probability in the time interval $[t, t+1)$ is

$$Pr(t \le T < t+1 | X_t = x_t; \mu_t, \xi_t) = \int_0^1 \frac{|\ln x_t + a_t|}{\sqrt{2\pi s^3}\sigma_K} \exp\left\{-\frac{(\ln x_t + a_t)^2}{2s\sigma_K^2}\right\} ds \tag{46}$$

The density of $X_t$ is

$$f_{X_t}(x_t; \mu_t, \xi_t) = \frac{1}{\sqrt{2\pi t}\sigma_K} \exp\left\{-\frac{(\ln x_t - \ln x_0 - \sum_{i=1}^t a_i)^2}{2t\sigma_K^2}\right\} \tag{47}$$

[16] A minor issue is that I assume $t$ can be infinity here. However, if we impose the upper bound of $t$, e.g. the maturity term of the mortgage, the only thing we need to do is to add boundary conditions.

Therefore,

$$
\begin{aligned}
Pr(t \leq T < t+1; \mu_t, \xi_t) &= \int_1^{+\infty} P(t \leq T < t+1 | X_t) f_{X_t}(X_t = x_t) dx_t \qquad (48)\\
&= \int_1^{+\infty} \int_0^1 \frac{|\ln x_t + a_t|}{\sqrt{2\pi s^3}\sigma_K} \exp\left\{-\frac{(\ln x_t + a_t)^2}{2s\sigma_K^2}\right\} ds \\
&\quad \frac{1}{\sqrt{2\pi t}\sigma_K} \exp\left\{-\frac{(\ln x_t - \ln x_0 - \sum_{i=1}^{t} a_i)^2}{2t\sigma_K^2}\right\} dx_t
\end{aligned}
$$

Finally, the hazard rate is

$$
h_t = \lim_{\Delta t \longrightarrow 0^+} \frac{Pr(t \leq T < t + \Delta t | T \geq t; \mu_t, \xi_t)}{\Delta t} \qquad (49)
$$

In discrete time, we have

$$
\begin{aligned}
h_t &= Pr\left(t \leq T < t+1 | T \geq t; \mu_t, \xi_t\right) \qquad (50)\\
&= \frac{Pr(t \leq T < t+1; \mu_t, \xi_t)}{Pr(T \geq t; \mu_t, \xi_t)} = \frac{Pr(t \leq T < t+1; \mu_t, \xi_t)}{1 - Pr(T < t; \mu_t, \xi_t)}
\end{aligned}
$$

Q.E.D.

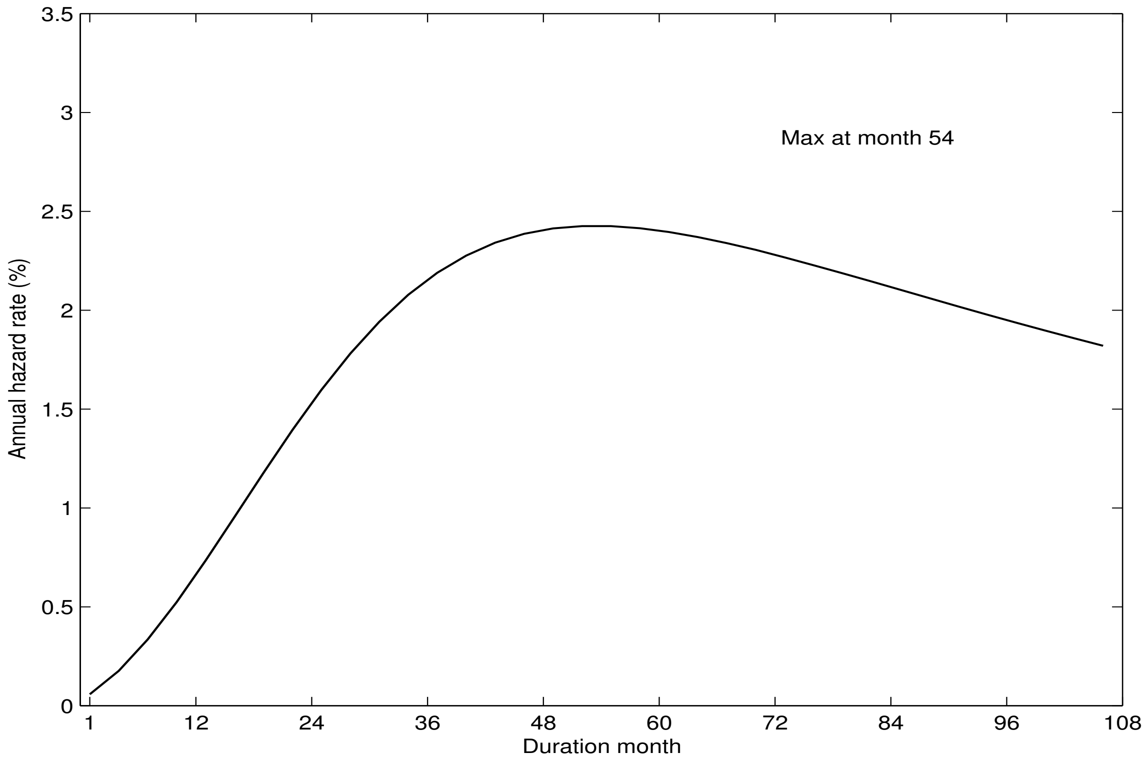


**Fig. 1. Estimated Seasoning Effect of Commercial Mortgage Loan Default**

This figure plots the hazard rates of a commercial mortgage over duration time, which shows the seasoning effect of default. The baseline parameters $\widehat{\nu}$ and $\widehat{\omega}$ are 0.065 and 2.610 respectively, and the covariates are at sample mean. The hazard rate reaches its maximum at 54 month (4.5 years). The 5-year cumulative default probability is 2.35%, and the 7-year cumulative default probability is 5.06% based on these estimates.

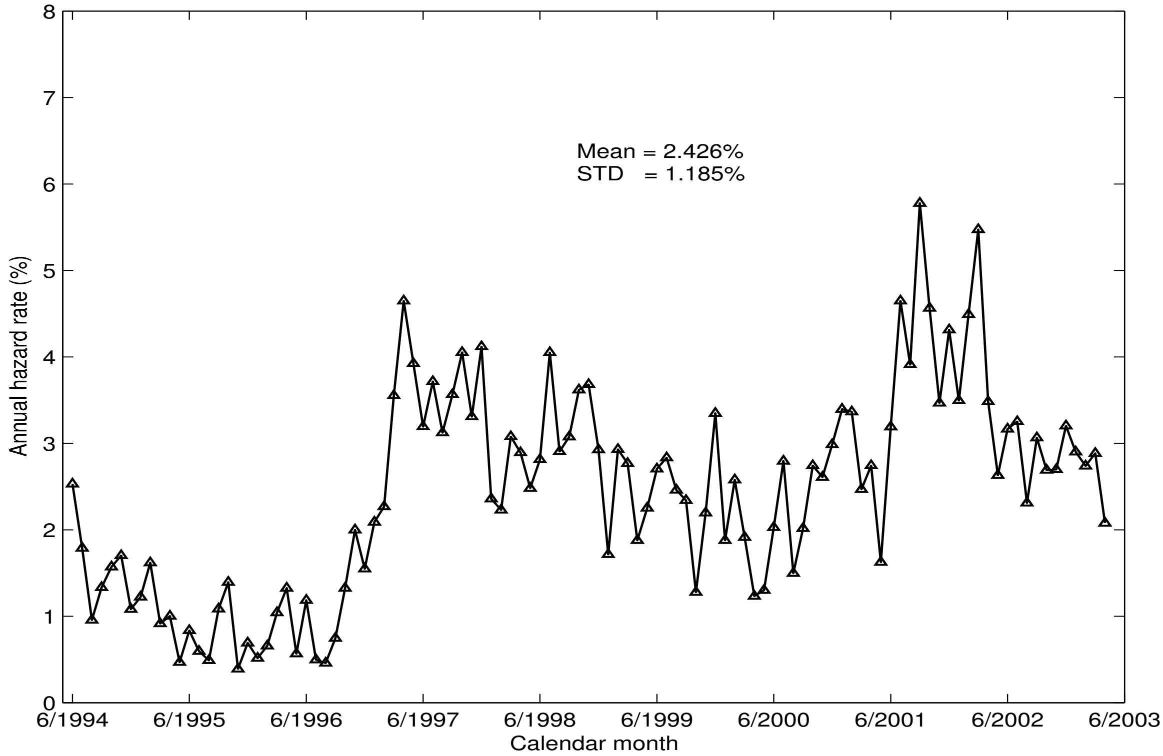


**Fig. 2. Estimated Hazard Rate Time Series of a Representative Mortgage**

The figure plots the hazard rates of a 4.5-year seasoned representative commercial mortgage over calendar time, which shows the time series dynamics of default. The hazard rate varies substantially over time from 16.11% of the sample mean to 238.20% of the sample mean.

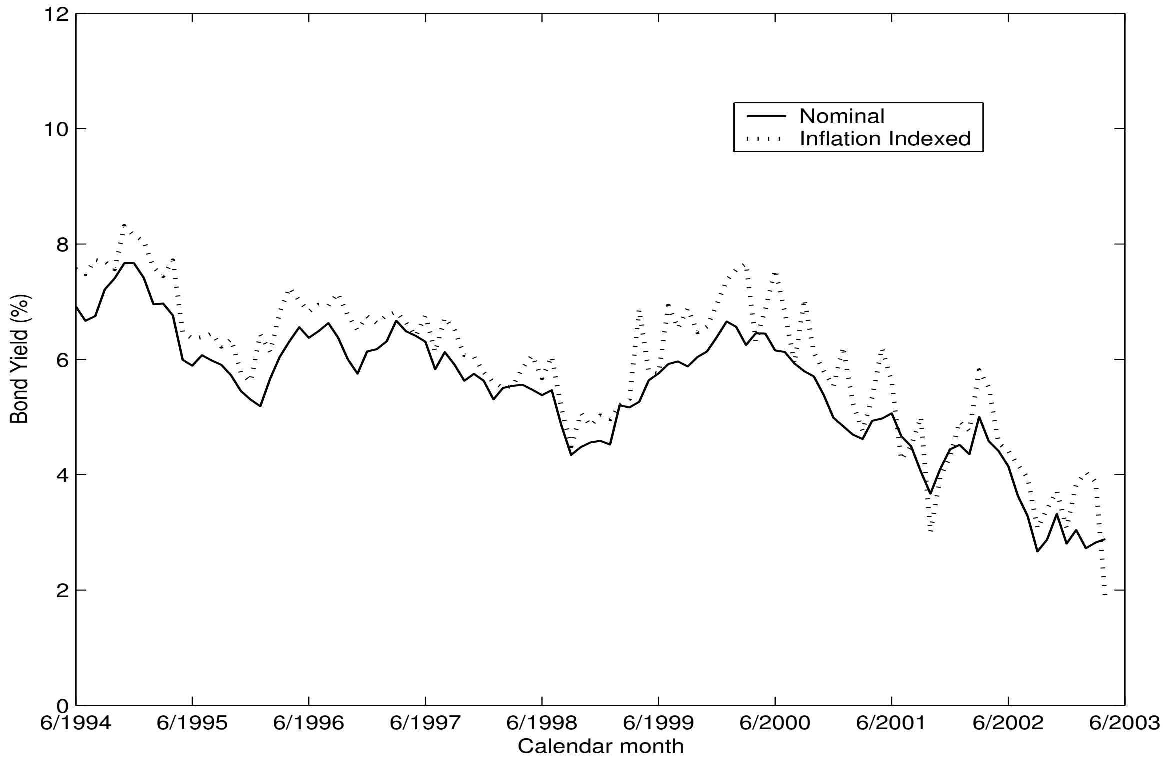


**Fig. 3. The Time Series of the 5-year Pure Discount Bond Yield**

This figure plots the Fama-Bliss (artificial) 5-year pure discount bond yields. The nominal yields are from CRSP, and CPI is used to make adjustments to obtain the real yields.

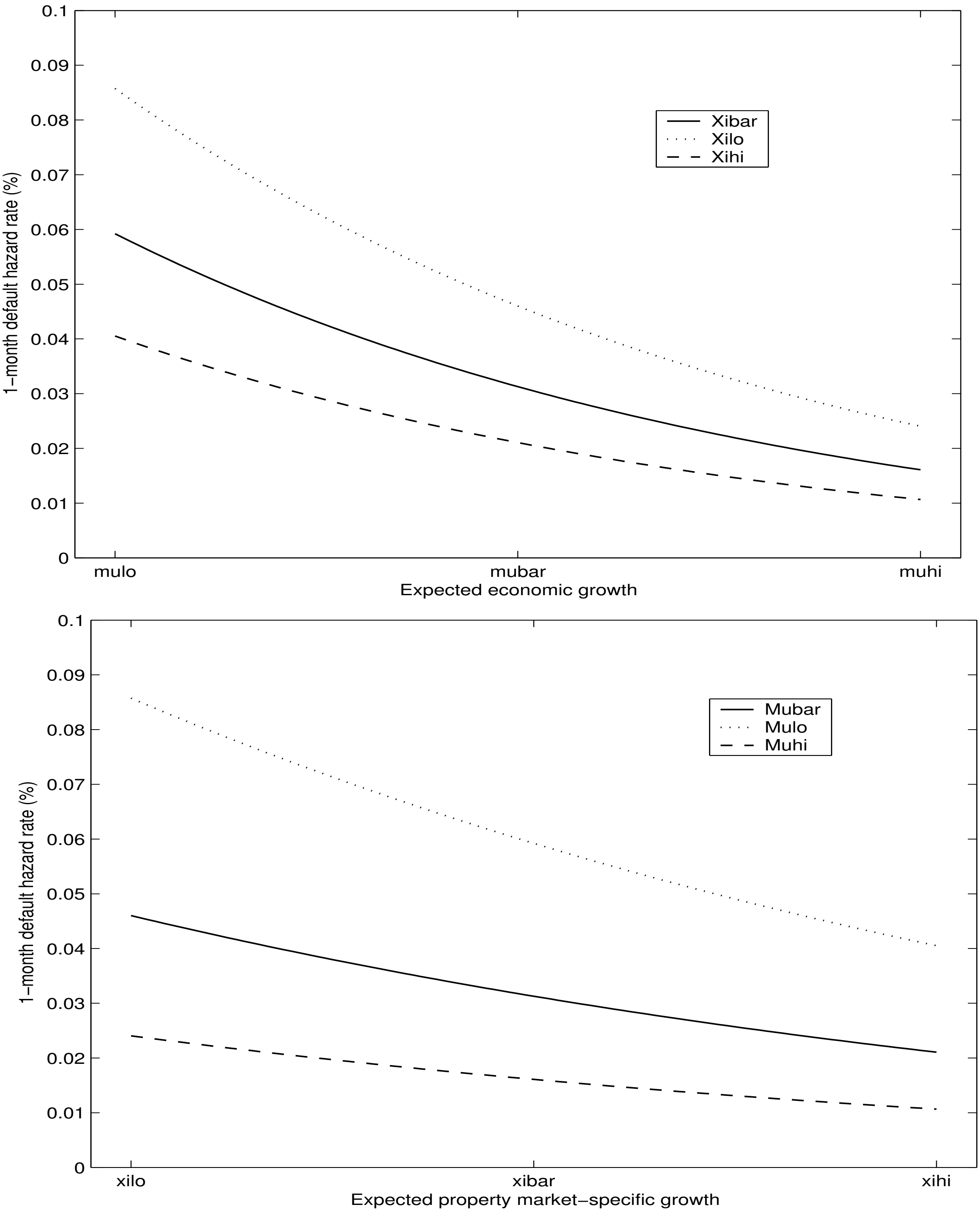


**Fig. 4. Simulated Relationship between Hazard Rate and State Variables**

**Table 1 Origination Timing of the Commercial Mortgage Loans**

This table shows the origination year distribution of 49,389 commercial mortgage loans in our study, which are underlying Commercial Mortgage-backed Securities (CMBS).

| Origination year | Number of loans | Percentage in # of loans | Total loan amount($m) | Percentage in loan amount |
|---|---|---|---|---|
| 1992 | 673 | 1.36 | 2,065 | 0.68 |
| 1993 | 1,214 | 2.46 | 5,324 | 1.76 |
| 1994 | 1,480 | 3.00 | 6,832 | 2.25 |
| 1995 | 2,151 | 4.36 | 9,699 | 3.20 |
| 1996 | 4,254 | 8.61 | 18,745 | 6.18 |
| 1997 | 6,472 | 13.10 | 36,552 | 12.05 |
| 1998 | 13,259 | 26.85 | 66,907 | 22.05 |
| 1999 | 5,660 | 11.46 | 36,001 | 11.87 |
| 2000 | 4,422 | 8.95 | 29,258 | 9.64 |
| 2001 | 4,906 | 9.93 | 45,439 | 14.98 |
| 2002 | 3,637 | 7.36 | 34,042 | 11.22 |
| 2003 | 1,261 | 2.55 | 12,515 | 4.13 |
| All years | 49,389 | 100.00 | 303,379 | 100.00 |

**Table 2 Characteristics of the Commercial Mortgage Loans**

This table shows the compositions of the commercial mortgage loans in the study by different loan characteristics, and percentage of loans defaulted in each category. Other property type includes manufactured housing, self-storage, mixed-use, etc. The average loan-to-value ratio(LTV) is 68.21%, and the average original balance is $6.14 million.

| Characteristics | Category | Number of loans | Percentage in # of loans | Proportion defaulted (%) |
|---|---|---|---|---|
| Property type | Multifamily | 16,052 | 32.50 | 2.69 |
| | Retail | 13,181 | 26.69 | 3.85 |
| | Office | 7,114 | 14.40 | 2.45 |
| | Industrial | 4,471 | 9.05 | 2.77 |
| | Hotel | 2,382 | 4.82 | 17.59 |
| | Healthcare | 975 | 1.97 | 12.82 |
| | Other | 5,214 | 10.56 | 1.84 |
| Region | Midwest/Eastern | 5,175 | 10.48 | 4.70 |
| | Midwest/Western | 1,950 | 3.95 | 4.92 |
| | Northeast/Mid-Atlantic | 5,962 | 12.07 | 3.52 |
| | Northeast/New-England | 2,807 | 5.68 | 3.28 |
| | Southern/Atlantic | 9,603 | 19.44 | 4.44 |
| | Southern/East-coast | 1,806 | 3.66 | 6.98 |
| | Southern/West-coast | 6,266 | 12.69 | 4.82 |
| | Western/Mountain | 4,642 | 9.40 | 3.79 |
| | Western/Northern Pacific | 4,784 | 9.69 | 2.15 |
| | Western/Southern Pacific | 6,394 | 12.95 | 1.61 |

**Table 2 (continue) Characteristics of the Commercial Mortgage Loans**

This table shows the compositions of the commercial mortgage loans in the study by different loan characteristics, and percentage of loans defaulted in each category. Other property type includes manufactured housing, self-storage, mixed-use, etc. The average loan-to-value ratio(LTV) is 68.21%, and the average original balance is $6.14 million.

| Charact-eristics | Category | Number of loans | Percentage in # of loans | Proportion defaulted (%) |
|---|---|---|---|---|
| Amortization term | less than 10 years | 1,310 | 2.65 | 2.37 |
| | 10-20 years | 7,261 | 14.70 | 5.45 |
| | 20-30 years | 39,313 | 79.60 | 3.61 |
| | greater than 30 years | 1,505 | 3.05 | 2.00 |
| Maturity term | less than 5 years | 2,563 | 5.19 | 2.69 |
| | 5-10 years | 36,780 | 74.47 | 3.61 |
| | 10-15 years | 5,320 | 10.77 | 4.72 |
| | greater than 15 years | 4,726 | 9.57 | 4.82 |
| Loan-to-value ratio | less than 40% | 1,853 | 3.75 | 1.46 |
| | 40-60% | 6,995 | 14.16 | 4.07 |
| | 60-80% | 38,482 | 77.92 | 3.86 |
| | greater than 80% | 2,059 | 4.17 | 3.84 |
| Original balance | less than $0.35m | 2,471 | 5.00 | 2.23 |
| | $0.35m-$18.25m | 44,447 | 90.00 | 3.98 |
| | greater than $18.25m | 2,471 | 5.00 | 2.23 |
| All loans | | 49,389 | 100.00 | 3.80 |

**Table 3 Maximum Likelihood Estimates of the Hazard Model**

This table presents the maximum likelihood estimates of the $\beta$ coefficients of the hazard model $h_i(t, \tau) = h_0(\tau) \exp(D_{i,t}\alpha + Z_i\beta)$. The baseline function $h_0(T) = \frac{\nu\omega(\nu T)^{\omega-1}}{1+(\nu T)^{\omega}}$ is estimated jointly with the covariate parameters, and is plotted in figure 1. The $\alpha$ coefficients of time dummies are presented in figure 2. The reference groups for loan-to-value ratio (LTV), property type, region, amortization term, maturity term and original balance are 60-80%, retail, Northeast/Mid-Atlantic, 20-30 years, 5-10 years and 0.35-18.25 million respectively. The asymptotic covariance is calculated with the information matrix using the BHHH method.

| Covariates | Estimate | Asy. S.E. | T stat. |
|---|---|---|---|
| Loan-to-value ratio (LTV) $\leq$ 40% | -0.030 | 0.080 | -0.372 |
| Loan-to-value ratio (LTV) 40-60% | 0.045 | 0.045 | 0.987 |
| Loan-to-value ratio (LTV) $>$ 80% | 0.159 | 0.073 | 2.182 |
| Multifamily | 0.212 | 0.047 | 4.547 |
| Office | -0.025 | 0.068 | -0.372 |
| Industrial | 0.017 | 0.070 | 0.248 |
| Hotel | 1.210 | 0.057 | 21.160 |
| Healthcare | 0.988 | 0.078 | 12.692 |
| Other | 0.035 | 0.071 | 0.491 |
| Midwest/Eastern | 0.199 | 0.063 | 3.161 |
| Midwest/Western | -0.048 | 0.089 | -0.544 |
| Northeast/New-England | -0.019 | 0.074 | -0.262 |
| Southern/Atlantic | 0.076 | 0.060 | 1.268 |
| Southern/East-coast | 0.218 | 0.081 | 2.673 |
| Southern/West-coast | 0.225 | 0.064 | 3.500 |
| Western/Mountain | -0.007 | 0.075 | -0.097 |
| Western/Northern Pacific | -0.362 | 0.083 | -4.354 |
| Western/Southern Pacific | -0.539 | 0.085 | -6.343 |
| Amortization term $\leq$ 10 years | 0.484 | 0.070 | 6.949 |
| Amortization term 10-20 years | 0.334 | 0.047 | 7.055 |
| Amortization term $>$ 30 years | -0.105 | 0.165 | -0.640 |
| Maturity term $\leq$ 5 years | 2.531 | 0.052 | 48.958 |
| Maturity term 10-15 years | -0.546 | 0.072 | -7.569 |
| Maturity term $>$ 15 years | -0.808 | 0.075 | -10.745 |
| Original balance $\leq$ 0.35 m | 0.474 | 0.060 | 7.873 |
| Original balance $>$ 18.25 m | -0.425 | 0.103 | -4.118 |

**Table 4 Estimates of the State Space Model**

This table presents the extended Kalman Filter estimates of the state space model. The processes of the two unobservable state variables, expected economic growth rate and the expected commercial property market-specific growth rate, are

$$d\mu_t = \kappa(\overline{\mu} - \mu_t)dt + \sigma_\mu dW_t^e$$

$$d\xi_t = \lambda(\overline{\xi} - \xi_t)dt + \sigma_\xi dW_t^K$$

and the measurement variables, bond yield $R_t$ and default hazard rate $h_t$ are given in equations (12) and (22). For identification purposes, I pre-specify the volatilities of economic growth growth and property income growth $\sigma_e$ and $\sigma_K$ of 8%, the risk aversion parameter $\gamma$ of 4 and subjective discount factor $\delta$ of 4%. For bond yield, observation error is assumed to be zero. The parameters and state variables are jointly estimated with the extended Kalman Filter. The asymptotic covariance is calculated with the information matrix using the BHHH method. The root-mean-square error (RMSE) is between the fitted hazard rates and the actual hazard rates.

| Parameters | Estimate | Asy. S.E. | T stat. |
|---|---|---|---|
| The economic factor | | | |
| $\overline{\mu}$ | 0.0170 | 0.0063 | 2.6984 |
| $\kappa$ | 0.4572 | 0.1098 | 4.1639 |
| $\sigma_\mu$ | 0.0073 | 0.0002 | 36.5000 |
| The property market factor | | | |
| $\overline{\xi}$ | 0.0120 | 0.0053 | 2.2642 |
| $\lambda$ | 0.5962 | 0.0205 | 29.0829 |
| $\sigma_\xi$ | 0.0059 | 0.0009 | 6.5556 |
| $x_0$ | 1.4601 | 0.0325 | 44.9262 |
| $\beta$ | 0.5980 | 0.0312 | 19.1667 |
| Default series fitting | | | |
| $100 \cdot \sqrt{\Sigma_{22}}$ | 0.0030 | 0.0013 | 2.3077 |
| $100 \cdot RMSE$ | 0.0082 | | |
| $100 \cdot Sample\ mean$ | 0.0578 | | |

**Table 5 Default Hazard Rates Under Different Scenarios**

This table presents default hazard rates of a representative commercial mortgage with various seasoning under different scenarios based on my model estimates. The rates are annual, and in percent.

| Seasoning | Optimistic | Long term mean | Pessimistic |
|---|---|---|---|
| 1 month | 0.0313 | 0.0617 | 0.1181 |
| 1 year | 0.2836 | 0.5590 | 1.0696 |
| 3 year | 1.1249 | 2.2174 | 4.2431 |
| 5 year | 1.3070 | 2.5765 | 4.9302 |
| 7 year | 1.1612 | 2.2890 | 4.3800 |