Private Equity Incentive Contracting and Fund Leverage Choice When Investors Target Returns[1]

Timothy J. Riddiough
University of Wisconsin – Madison

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Abstract

We show how the standard PE compensation contract is used to create incentives for GPs to utilize leverage so that LP investors can meet their return targets. A theory of fund capital structure is developed in which investors trade off alpha with costs of financial distress. We then show how carried interest is used to fine-tune leveraging incentives, where there is a one-to-one mapping between the carried interest return hurdle and fund leverage. The fixed asset management fee and promote percentage are used to ensure fees meet or exceed the minimum fee required for GP participation. When costs of financial distress are sufficiently large relative to alpha, limits will exist on the ability to leverage a fund to meet LP return targets. Three different fee regimes are considered to analyze net-of-fee PE returns, where we show that fees generally increase incentives to leverage the fund. This analysis highlights pension fund investment behavior when there is a focus on absolute returns without close reference to risk considerations.

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I.   Introduction

Private equity (PE) investment has become an increasingly important channel for resource allocation in the U.S. and abroad, and is expected to continue to gain prominence. Its effect on the real economy is not insignificant, as, in contrast to many types of hedge fund strategies that involve the secondary market trading of financial securities, PE generally involves investment in real assets in which financial, operational and governance engineering are brought to bear to affect the real investment productivity (Kaplan and Stromberg (2008)). Corporate buyouts, venture capital and real estate development funds are prominent examples of PE investment.

As PE investment has grown, so has research on a variety of PE investment topics. This research has produced a fascinating array of empirical findings that would seem to be at odds with the conventional wisdom regarding incentive contracting, risk-and-return relations and capital structuring. In this paper we seek to comprehensively address these issues by developing a model of PE incentive contracting and fund leverage choice when investors only care about meeting return targets.

For lack of a better word we will refer to the anomalous stylized empirical "facts" as puzzles. There are four such puzzles that motivate our analysis. Puzzle #1 follows from PE utilizing incentive contracts, executed between the fund sponsor/manager (GP) and fund investors (LP). PE is a business whose investment performance does not appear to directly depend on direct effort outcomes linked to incentives created in the compensation contract. Rather, variation in performance across PE fund managers is generally attributed to endowed skill levels that make certain managers inherently more productive than others. This difference between effort and skill is important. Endowed skill has been emphasized because of findings of persistence in manager performance over time and the fact that "indirect" compensation—compensation received in the future that follows in part from current fund performance—has

been found to be more important than direct compensation associated with currently managed funds.[2] Without variable effort production and with endowed skill, it is unclear why performance-sensitive compensation is part of the compensation contract, as agents could do just as well or possibly better with a fixed compensation contract.

Related to this is that there is little apparent variation in PE compensation contracts across fund managers and PE asset classes. The "2-20" contract is quite common, with GP's paid a two percent fixed compensation fee based on assets (invested capital) under management and 20 percent of profits that exceed a carried interest hurdle rate (Robinson and Sensoy (2013), Chung, Sensoy, Stern and Weisbach (2012), Metrick and Yasuda (2010)).[3] Interestingly, more variability has been documented in the carried interest hurdle rate, which has been found to mostly lie in the 6 to 10 percent range, but with effective rates that sometimes exceed 20 percent (Metrick and Yasuda (2010)). It is also worth noting that GP ownership percentages, which are often one percent of invested capital, but with some variation around the median, show no relation to fund performance (Robinson and Sensoy (2013)). As a consequence, it is hard to not only rationalize the existence of effort-based incentive contracts when effort effects are *de minimus* with respect to current compensation outcomes but also to rationalize contracts whose key parameter values vary so little across fund managers who differ significantly in their experience and expertise, and who manage assets that can vary tremendously in management intensity.

Puzzle #2 follows from the LP investor side. Institutional investors are by far the most important PE investors, and pension funds are the most important institutional investors. There is increasing evidence that pension funds fail to conform to the time-tested, supposedly rock-solid laws of finance that intimately link investment risk with investment return. Rather, pension funds seem to emphasize absolute returns, particularly *targeted returns*, with little or no explicit emphasis on risk or relative returns (Andonov and Rauh (2017)). Gompers, Kaplan and

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[2] See Lim, Sensoy and Weisbach (2016) for evidence from hedge funds, where in conclusion they state, "The lack of a relation between a hedge fund's contractual fee structure and indirect, market-based incentives reflects a larger puzzle about the structure of compensation in alternative asset classes." See Chung, Sensoy, Stern and Weisbach (2012) for evidence from PE.

[3] As Metrick and Yasuda (2010) relate: "The exact origin of the 20% focal point is unknown, but previous authors have pointed to Venetian merchants in the Middle Ages, merchant sea voyages in the age of exploration, and even the book of Genesis as sources."

Mukharlyamov (2016) report that institutional PE investors employ internal rate of return (IRR) and investment multiple methods when making investment decisions—methods to do not require risk measures as an input to the analysis—rather than NPV methods that require a risk-adjusted discount rate as an input to investment analysis.

The delinkage of risk and return in PE investment follows in part because so many public pension funds are significantly underfunded (Novy-Marx and Rauh (2009, 2011). This can cause funds to reach for yield, in that they target returns first and consider risk consequences later (Andonov and Rauh (2017)).[4] Pension fund regulation further muddles traditional risk-return sensitivities by allowing pension funds holding riskier investments to use higher discount rates when valuing future pension fund liabilities—liabilities whose risk characteristics may have little or nothing to do with asset risk characteristics (Brown and Wilcox (2009), Andonov, Bauer and Cremers (2017)). This regulation thus introduces perverse incentives to increase investment risk, which in turn reinforces tendencies to reach for yield. Further, Bodnaruk and Simonov (2016) document that underwater pension fund managers exhibit loss averse preferences, which actually encourages risk-seeking behavior (see also Wang, Yan and Yu (2014)).

Principal-agency in the pension fund world also points to delinkages in classic risk-return relations. "Prudent man" rules written into the original ERISA regulations of the 1970s suggest that if you fail in your investment strategy, you should fail with a great deal of company, implying a herd mentality with respect to investment allocation strategies (Scharfstein and Stein (1990)). Relatedly, pension fund investment managers, particularly those who are dealing with underfunding situations, are thought to be averse to *measured risk*. These managers exhibit preferences for opaque investments such as private equity and hedge funds that create difficulties in measuring current investment values, which subdues observed price volatility. Here risk obfuscation benefits investment managers by avoiding the *recognition* of bad investment outcomes over short horizons.

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[4] Axelson, Sorensen and Stromberg (2014) express the conundrum a different way, and refer to it as the "β puzzle" as they ponder the following: "These studies suggest that buyout funds can acquire regular companies with equity β around 1.0 and then increase their leverage six-fold, yet leave systematic risk unchanged." Also see Pagliari (2017) and Bollinger and Pagliari (2019) for evidence for performance misalignment of PE real estate funds in risk and return space. For broader evidence on the reaching for yield phenomena, see Becker and Ivashina (2015) as applied to insurance company investment, DiMaggio and Kacperczyck (2017) who analyze money market funds and Choi and Kronlund (2018) for evidence from corporate bond mutual funds.

This brings us to puzzle #3, which follows from John Cochrane's (2011) famous quip, "there is no alpha, only beta that we understand and beta that we don't understand." That may be, but persistence in fund performance, a belief that GP fund managers do possess certain investment skills that others do not have (founded on a notion of labor market incompleteness), and studies that show that PE generally delivers returns that exceed the market portfolio return by about 8 percent before fees and 4 percent after fees have led to an acceptance that positive alpha in fact exists and persists in the PE world.[5] Although its presumed existence is not uncontroversial, positive alpha is something that we don't dispute in this paper. Rather, similar to Lan, Wang and Yang (2013) in the context of hedge funds and Sorensen, Wang and Yang (2014) (hereafter SWY) with PE, we posit the existence of positive alpha, and then exploit its properties to help address related issues.

Finally, puzzle #4 relates to the existence of "cheap debt" that is supposedly available and unique to PE investment, particularly buyout funds, which tend to use generous amounts of leverage. Earlier literature often referred to PE debt as "favorably priced" and "mispriced," where more recently SWY (2014) have clarified the notion of what they call "cheap debt."[6] Cheap debt is made available by lenders that are well diversified and have access to low cost funds, and that recognize GP fund managers are capable of generating positive alpha. Low debt funding costs, positive alpha, and no financial distress costs combine to create cheap debt as benchmarked against a higher cost of equity capital. The result is fund leverage that "maxes out" by hitting an exogenously imposed debt ceiling, implying a fund capital structure that is vastly different from run-of-the-mill corporations.[7]

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[5] See Harris, Jenkinson and Kaplan (2014) and Gompers, Kaplan and Mukharlyamov (2016). Franzoni, Nowak and Philippou (2012) find that, net cash flows from PE investment are lower when credit conditions are tightening and when liquidity conditions are deteriorating, suggesting that liquidity risk should load positively as a priced risk factor. Although we do not incorporate liquidity risk explicitly into our model, losses that occur due to borrower default and reselling the repossessed collateral, which occurs more frequently and at high loss levels when market conditions are weak and the marginal propensity to consume is high, can adjust in anticipation of future market conditions.

[6] See also Axelson, Jenkinson, Stromberg and Weisbach (2013).

[7] As noted by Axelson, Stromberg and Weisbach (2009) in reference to the risk attitudes of GPs who lever up on behalf of their investors: "Practitioner: Ah yes, the M-M theorem. I learned that in business school. We don't think that way at our firm. Our philosophy is to lever our deals as much as we can, to give the highest returns to our LP's."

This paper comprehensively addresses these apparent puzzles by developing a simple model of PE fund financing in which leverage is used to meet LP return targets.[8] Our model uses a simplified version of the SWY (2014) model as a starting point, where debt in our model incorporates cheap funding costs and positive alpha. But in addition we consider costs of financial distress. This creates a tradeoff between positive alpha (a negative market friction that favors debt) and costs of financial distress (a positive market friction that disfavors debt) that is new to the capital structure and PE literature.[9]

In a first-best world that puts aside incentive contracting between the LP and GP, we show that there exists a unique internal leverage optimum that characterizes the optimal capital structure of PE fund investment. In addition to the tradeoff between alpha and financial distress costs, expensive equity capital costs may exist, which tilts debt levels higher.[10] Our analysis further clarifies what is meant by the phrase "cheap debt," where we show that expensive equity capital is necessary for positive leverage to dominate zero leverage in the presence of positive alpha. That is, positive alpha by itself is not sufficient to generate optimal levered capital structures—expensive equity is required. Without expensive equity, M-M irrelevance results occur when financial distress costs are zero, and zero debt is optimal when financial distress costs are positive. However, positive alpha in the presence of expensive equity magnifies the attractiveness of debt in the capital structure.

We then pose the question of how leverage can be used at the fund level to meet the return targets of PE investors. We show that the observed carried interest incentive compensation contract is sufficient and sometimes necessary to generate the desired outcomes. In other words, we argue that incentive contracting practices observed in PE have little to do with operational or

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[8] In their analysis of PE real estate funds, Shilling and Wurtzbach (2012) find, "Managerial compensation packages, which provide strong incentives for core property managers to take on more leverage (within limits) in order to achieve a 'target' rate of return, especially when property prices are rising and yields are low. In contrast, we generally find that value-add and opportunistic funds [which are riskier than core funds] consistently use high leverage regardless of market conditions."

[9] As noted by Berk and DeMarzo (2017) in their textbook on corporate finance, "When securities are fairly priced, the original shareholders of a firm pay the present value of the costs associated with bankruptcy and financial distress."

[10] The notion of expensive equity originates from the pecking order model of Myers and Majluf (1984). The recent capital structure literature takes two complementary approaches to modeling expensive equity. Whited (XX) represent models of equity cost by taking a haircut at the time of issuance without adjusting the equity discount rate, whereas DeMarzo represent an approach in which expensive equity is baked into a higher discount rate than that applied by more patient lenders who price the debt.

governance engineering, but rather are financing engineering mechanisms used to incentivize the GP to lever the fund appropriately to generate returns that meet the investor's target.[11] We demonstrate a one-to-one mapping between leverage and a specified carried interest return hurdle, where the optimal contract is free of GP return preferences and does not require the carried interest percentage parameter. This latter result implies the carried interest incentive contract observed in practice is over-identified in terms of inducing incentive compatible outcomes between the GP and LP.

While the LP uses the carried interest hurdle rate to incentivize fund leverage chosen by the GP, the GP's objective is to satisfy a total fee requirement. Total fees therefore function as a participation constraint that must be met by the LP to ensure the GP will commit to fund management. With this the contracting process is complete, where first the LP identifies fund leverage that satisfies target return objectives. The LP then maps the required leverage into a carried interest hurdle return that incentivizes the GP to leverage the fund appropriately on behalf of the investor. The carried interest and fixed fee percentages are then determined by the LP to satisfy participation requirements of the GP, with the full set of contract parameters written into the compensation contract prior to the start of the fund's life.

Before considering net-of-fee investment returns, we characterize investment return properties on a gross-of-fee basis. When debt is sufficiently cheap, in the sense that financial distress costs are small relative to alpha, fund returns increase without bound with continued increases in leverage, albeit at a slower rate than when financial distress costs are zero or very small. This result confirms and generalizes findings of SWY (2014), where we extend their model to incorporate costs of financial distress. However, when financial distress costs are sufficiently large relative to alpha, gross-of-fee returns reach an internal maximum as a function of leverage and then turn downward. This result further generalizes previous findings and demonstrates limits to leverage in meeting fund return targets.

Finally we consider net-of-fee returns under three different fee regimes: i) fixed fee and carried interest percentages that are set exogenously at industry "norms"; ii) the carried interest

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[11] We are not the first to ponder the PE incentive contracting-fund capital structure question. In contrast to our approach that emphasizes LP return targeting, Axelson, Stromberg and Weisbach (2009) argue that there is an operational/governance connection to using liberal amounts of debt to reduce agency conflicts between GPs and LPs, in which debt is cross-collateralized by all of the assets of the fund.

percentage is used exclusively to satisfy the GP's participation constraint; and, iii) the GP extracts fees in proportion to the alpha it generates for investors.[12] Results indicate GP compensation contracting parameters that are in line with those observed in practice. For example, the carried interest return hurdle is generally between 5 and 20 percent for realistic parameter value constellations, which compares with the 6 to 10 percent that is often observed in practice. Total fees and the breakdown between fixed and variable fees are also generally in line with prior empirical estimates.

The larger point of this paper is to shine a light on perhaps the most disturbing of the four highlighted anomalies: underfunded pension funds that reach for yield by targeting high returns from PE investment, in which considerable leverage is utilized in the process.[13] Implications of such behavior include exposing taxpayers, if not pensioneers, to the consequences of failed investment strategies, which could impose massive intergenerational social costs. Such behavior further potentially introduces significant distortions into the pricing of real assets, as the demand for those assets has accelerated in recent years.[14] For example, the pricing of commercial real estate in the premier international markets such as New York City, London, Tokyo and Hong Kong, has exploded in recent years with increasingly concentrated institutional holdings in those cities.[15] Sustained negative economic shocks to those areas will eventually test the wisdom of such investment policies, which are premised on stability of the local economies and the associated liquidity that goes with investment in premier international economic hubs.

The greatest unknown risks associated with pension fund PE investment are the systemic implications of these leveraging strategies. To date the systemic risks of institutional investment in PE have been played down, but PE is being increasingly financing by shadow banks that include funding vehicles such as collateralized loan obligations (CLOs) that are, in turn, often

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[12] The third regime in which fees are positively associated with alpha can be motivated by Robinson and Sensoy's (2013) finding that higher fees are associated with better gross-of-fee performance, with essentially no relation between fees and net-of-fee performance.

[13] For a much more positive view on the benefits of pension systems on financial market development, see Scharfstein (2018).

[14] Andonov, Bauer and Cremers (2017) observe, "given the amount of assets under management by pension funds, correlated changes in their strategic asset allocation could also have implications for asset pricing."

[15] According to November 2017 data obtained from Lasalle Investment Management, 48 percent of all office, warehouse and retail commercial property in New York City is institutionally owned, compared to 66 percent in Tokyo, 69 percent in Hong Kong and 74 percent in London. These are surprisingly high, almost scary concentrations, but we note that not all of this ownership will be classified as held by PE funds.

equity financed by institutional investors. This funding circularity introduces obvious moral hazard problems, which combined with a lack of transparency into the detailed investment activities of most institutional investors, bears an eerie resemblance to what we did not know that we did not know about repo finance and the mortgage-backed securities markets the last time around.

The organization of the paper is as follows. In the next section we outline the model structure. Section III introduces our model of debt pricing that incorporates alpha as well as costs of financial distress. Working backwards by induction, in section IV we consider incentive fees and the GP's leverage choice problem. Then in section V we analyze the LP's contracting problem when LP's target returns and when total fees paid to the GP establish a participation constraint.

II. Model Structure

The assets that populate a PE fund are "real assets" that contain, or embody, the traditional factors of production – land, labor and physical capital – or some combination of the three. This definition of real asset investment covers the main categories of PE that include buyout (BO), venture capital (VC) and commercial real estate (CRE) funds.

With PE ownership, representative asset managers are highly skilled. By this we mean that they are capable of generating superior value over time (positive alpha). As emphasized by Kaplan and Stromberg (2008), manager skill combined with unique aspects of the PE investment vehicle itself can allow for operational, governance and financial engineering that results in real value creation that is exclusive to the fund. Consistent with the approach of SWY (2014) and many others, we will take alpha as given. We then examine the consequences of positive alpha on incentive contracting and leverage choice as well as on fund returns. Throughout we will refer to these PE fund managers as GP's.

Although highly skilled, PE asset managers lack capital. This causes them to partner with external equity investors, which we label as LP's, to provide the necessary equity capital. Outside debt may also be sourced by PE fund manager, resulting in three distinct agent-sectors to be analyzed: 1) The GP that manages the fund by making the investment, operating and

financing decisions, 2) The LP that invests equity capital into the fund, and 3) The lender that provides debt financing on a limited liability basis. In contrast to the GP, the LP has no asset management skills, only available capital which it can deploy to invest. The lender provides debt financing, at a competitive price that reflects the associated credit costs and risks. The LP and the lender, given their respective provisions of equity and debt capital, have their own set of investment objectives. Given its superior management skills, the GP in its role as fund manager wants to be appropriately compensated for its services.

There are three relevant contracts to consider in this PE fund setting (see Kaplan and Stromberg (2008) for additional background). First, there is the PE *fund investment contract* that is drawn up by the GP and offered to the LP. This specifies the asset class to be targeted for investment and the total equity capital to be raised by the fund. The fund investment contract also specifies the life of the fund. The second is a *compensation contract* that specifies how and how much compensation is to be paid by the LP to the GP for fund management service provision. This contract is offered by the LP to the GP, typically after a negotiation over total fees. From a compensation perspective, the GP in our model is focused on total fees, without any particular concern over the split between fixed and variable compensation. This is consistent with findings that show that reputation effects and indirect compensation dominate direct incentive-based compensation tied to the current fund performance (see, e.g., Robinson and Sensoy (2013), Lim, Sensoy and Weisbach (2016)). The third contract is the *debt contract* that is executed between the GP and the lender, with the GP representing the interests of the LP in negotiating the terms and conditions of the contract.

The quantity of equity raised is in accordance with the fund investment contract. Then, based on the incentive compensation contract that is offered by the LP to the GP, the GP chooses how much leverage to use to finance investment. Debt funding is not released until investment occurs, since the to-be-acquired real assets provide security for the loan. Because the PE fund investment contract restricts investment to a particular class of real assets, and because the debt is raised only at the time of investment in response to incentives created by the compensation

contract, the investment-financing problem is a standard one of investment opportunities in search of an optimal funding structure.[16]

Equity pooling, real asset investment and debt financing occur in the order described, but for modeling purposes are compressed to a single point in time, time $t=0$, the start date of the fund.[17] Given this process, we proceed by working in reverse order from the lender's problem of pricing the debt, to the GP's problem of pinning down leverage and total assets under management conditional on the incentive compensation contract being offered by the LP, and finally to the LP's problem of meeting its investment return objectives by setting out the terms of the GP's incentive compensation contract.

For modeling purposes we treat all real assets that populate a fund as one large asset. In practice, PE fund offering documents will typically restrict investment into one or a few related industries in the case of buyout funds, or into only certain types of similar-risk commercial real estate in the case of real estate PE funds. These standard investment parameters imply that asset diversification within a fund is not a primary LP investment objective, and hence not of first-order modeling importance.

Our model structure has strong similarities to that of SWY (2014). A key difference is our focus on the endogeneity of the GP compensation contract, whereas in SWY the compensation contract is exogenously specified. SWY consider an investment environment with and without liquidity risk, whereas we abstract from liquidity risk to focus on optimal contracting and capital structure decisions. In both models debt financing is available from competitive, well diversified and patient lenders with access to cheap funding who price the debt after accounting for the risks of default. A critical departure from SWY is that we augment the

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[16] If capital is committed by creditworthy LP's prior to investment, where the GP subsequently pledges the committed capital as security to a debt provider prior to investment, the classical investment-financing model looks more like one of capital in search of investment. Even then, however, there will be limits on leverage imposed by secured lenders, as well as on the GP-LP contract terms. See Axelson, Stromberg and Weisbach (2009) for more on this issue, where they note, "Typically these [PE] funds raise equity capital at the time they are formed, and raise additional capital when the investments are made… [where] this additional capital usually takes the form of debt when the investment is collateralizable."

[17] We therefore abstract from issues associated with committed versus invested capital, among others. Metrick and Yasuda (2010) provide a refined definition of committed capital as equaling invested capital plus lifetime fees plus establishment cost. We ignore establishment cost and assume fees are paid out of pocket at the time incurred, without a committed capital set-aside. See Arnold, Ling and Naranjo (2017) for an empirical examination of committed funds that are waiting to be called. Also see Arcot, Fluck, Gaspar and Hege (2015) for evidence of how "time pressured" PE fund buyers and sellers perform relative to less pressured buyers and sellers.

model to incorporate costs of financial distress, which introduces limits on the availability of "cheap" debt with which to fund positive alpha investment projects.

Fund asset value, $V_{i,t}$, is modeled to evolve stochastically according to geometric Brownian motion, characterized by constant drift parameter, μ_i, and constant volatility parameter, σ_i. Real assets available for investment fall into discrete categories or classes, as identified by the subscript i, $i \in [1, N]$, with the N asset classes ordered from least to most risky. Within each asset class, there is a pairing, (α_i, σ_i), that fully characterizes real asset investment characteristics. $\mu_i = r + \alpha_i$, $\alpha_i \geq 0$, determines the drift rate of asset values in asset class i as held by a skilled PE fund manager and $\sigma_i > 0$ determines the volatility of asset values in the asset class. We might expect riskier asset classes to be associated with higher alphas, since managing those risky asset classes will generally require greater skill. That said, because relatively little is known empirically about the relation, we will not impose any particular functional form with respect to how asset class risk maps into expected return.

Based on the GP's skill and experience, and as outlined in the fund's offering documents, the GP invests in one and only one asset class at a time. As a result, LP equity investor clienteles will match with fund asset categories offered by GP's. The match in our model is determined by whether target investment return objectives of the LP can be met by the combination of investment in a particular asset class and the endogenous fund leverage choice.

All agents in our model are risk neutral, which is a common assumption in the PE and hedge fund literature (e.g., Axelson, Stromberg and Weisbach (2009), Panageas and Westerfield (2009), Lan, Wang and Yang (2013)). It is also a common assumption in the optimal contracting and capital structure literature, which is the primary focus of this paper (e.g., DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007)). When, as modeled by SWR (2014), markets are incomplete but state-contingent outcomes (risks) are fully spanned by an existing set of investment opportunities, alpha can persist and PE contingent-claim pricing can occur as if investors are risk neutral, implying little loss of generality. We do allow for the GP and LP to exhibit impatience, however, with a discount rate that may exceed r.[18] In the case of the GP we will denote this discount rate as γ_i, $\gamma_i \geq r$. LP equity investors express their temporal return

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[18] Also see, e.g., DeMarzo (2005) and DeMarzo and Fishman (2007).

preferences through their return targets. We will denote targeted return as λ_j, $\lambda_j > r$, where the j subscript identifies LP investor j. LP investment return objectives are determined independently of asset class characteristics. This results in LP investor j matching itself to fund managers of asset classes that are capable of meeting targeted net-of-fee return objectives.

III.   Debt Pricing with Alpha and Costs of Financial Distress

The Debt Value Function

We now develop a model of debt financing for PE fund investment. Recall the time t value of assets from asset class i equals $V_{i,t}$, where $t=0$ indicates the start date of the fund as well as date of debt issuance. Debt term for fund i matches the stated fund life, T_i. Although exogenous in our model, we recognize that PE funds vary in their maturities, oftentimes as a function of the type and risk of the fund. The collateralized loan is structured as balloon, or zero coupon debt, with recourse only to the fund's assets in case of default. As in Merton (1974), default will only occur at the maturity date, T_i, happening only if the asset value of the fund at that time, $V_{i,T_i} < B_i$, where B_i is the face amount of the debt due at maturity. To streamline the analysis, and with no loss of analytical richness, we will not consider any strategic bargaining that could occur between the lender and the fund investor over the deadweight costs incurred by the lender as a result of investor default.[19]

The deadweight costs incurred by the lender in the case of PE fund default are proportional to V_{i,T_i}, with the cost parameter denoted by k_i, $0 \leq k_i \leq 1$. We allow these costs to vary across asset classes, since assets belonging to riskier asset classes, possibly with higher alphas, are likely to be more difficult to manage given fewer skilled industry experts available to purchase the repossessed assets and operate them efficiently (Shleifer and Vishny (1992)). For similar reasons the cost parameter may vary depending on the real assets' production ratio of human capital to physical capital or land. For example, commercial property funds, especially the lower risk funds, are mostly physical capital and land, implying more tangible collateral and higher recovery rates. In contrast, VC is often almost exclusively highly specific human capital,

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[19] If *ex post* bargaining could occur to reduce the payoffs to the lender, the lender would anticipate this *ex ante* and increase the cost of the debt to compensate for bargaining effects.

where failure results in little or no recovery value. Buyout funds typically fall in between, where the most highly levered buyouts tend to be associated with firms that are rich in land and physical assets, but have little or no going concern value tied to human capital.

Recalling that real asset prices to PE investment evolve stochastically, following geometric Brownian motion with parameters μ_i and σ_i, debt value at the time of issuance equals $D_{i,0}$, determined as follows:

$$D_{i,0} = e^{-rT_i}\mathbb{E}_0\big[Min\{\tilde{V}_{i,T_i}{}^\wedge(1-k_i)\tilde{V}_{i,T_i}, B_i\}\big] \tag{1}$$

where default occurs when $V_{i,T_i} < B_i$, with the lender recovering $(1-k_i)V_{i,T_i}$ conditional on default.

Equation (1) can be reexpressed as,

$$D_{i,0} = e^{-rT_i}\left[B\int_{B_i}^{\infty} f(\tilde{V}_{i,T_i}|V_{i,0})d\tilde{V}_{i,T_i} + (1-k_i)\int_{B_i}^{\infty}\tilde{V}_{i,T_i}f(\tilde{V}_{i,T_i}|V_{i,0})d\tilde{V}_{i,T_i}\right] \tag{1a}$$

where $f(\tilde{V}_{i,T_i}|V_{i,0})$ is the pdf for $\tilde{V}_{i,T_i}$ that corresponds to the equation of motion governing prices for assets in class i.

Solutions to the integrals are well known, expressed as follows:

$$D_{i,0} = e^{-rT_i}BN\big[d_{2,i}\big] + (1-k_i)V_{i,0}e^{\alpha_i T_i}N\big[-d_{1,i}\big] \tag{1b}$$

where $N[\cdot]$ denotes the cumulative standard normal distribution, with

$$d_{1,i} = \frac{ln\big[{}^{V_{i,0}}/_{B_i}\big] + \big(\mu_i + \frac{1}{2}\sigma_i^2\big)T_i}{\sigma_i\sqrt{T_i}}, d_{2,i} = d_{1,i} - \sigma_i\sqrt{T_i} \tag{1c}$$

Observe that debt value, $D_{i,0}$, is identical to the frictionless debt value analyzed in Merton (1974) when $k_i=0$ and $\mu_i = r$ (implying $\alpha_i=0$). This isolates the two deviations that exist in our model relative to the frictionless benchmark case. First, $k_i>0$ reduces debt value and hence proceeds relative to the benchmark case. Further, it is easy to see that as B_i becomes large, with a probability of default that approaches one, $D_{i,T_i} \to (1-k_i)V_{i,0}e^{\mu_i T_i}$, the post-liquidation asset value at the debt maturity date, T_i. Second, the drift term increases debt value when $\alpha_i>0$, as in SWY (2014)). This decreases the likelihood of default to increase debt proceeds.

As discussed by SWY (2014), positive alpha functions as a negative dividend, increasing rather than decreasing real asset value over time. Another interpretion of α_i is as a negative cost of carry. Cost of carry normally accrues to the benefit of a forward contract holder, since there is an advantage to owning a claim on an asset without actually having to pay the positive carrying cost that goes with ownership. In our case, the assetholder receives the benefit of carry in the form of positive alpha. With either interpretation, alpha functions as a negative friction, the result of unique and highly specific human capital that is endowed to a fund manager when labor markets are incomplete.

A key comparative static result derives from the relation between debt value and total leverage when there are costs of financial distress. The following lemma summarizes the result, which we refer to as the *choke condition*, along with uniqueness that follows from a single-crossing property.

Lemma 1 (The "Choke" Condition): $\frac{\partial D_{i,0}}{\partial B_i} = e^{-rT_i}\left[N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}}\right]$. **For any $k_i>0$, there exists a unique B_i, $B^*_{k_i}$, which satisfies $N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}} = 0$. $\frac{\partial D_{i,0}}{\partial B_i} > 0$ and $\frac{\partial^2 D_{i,0}}{\partial B_i^2} < 0$ for $k_i>0$ and $B_i \in [0, B^*_{k_i})$.**

Proof: See Appendix

When $k_i=0$, the standard comparative static result obtains in which debt value is a positive function of B_i. However, for any $k_i>0$, there will always exist a finite leverage level, denoted as $B^*_{k_i}$, at which the comparative static switches signs to become negative. Figure 1 displays how the key relation, $N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}}$, varies as a function of B_i. At $B_i=0$, $N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}} = 1$. For $k_i>0$, $N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}}$ is decreasing up to and past the point at which $N[d_{2,i}] = n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}}$. After crossing the choke threshold, debt value begins to decrease with increases in the debt's face value. A functional minimum will exist, after which the slope of the function turns positive and asymptotes to zero from below as $B_i\rightarrow\infty$. This key relation will be

utilized extensively throughout the paper as we characterize capital structure outcomes associated with PE fund investment.

Figure 1 Here

First-Best

We now abstract from the GP-LP compensation contract to analyze "first-best." We put first-best (FB) in quotes, since there is a tradeoff to eliminating the GP-LP compensation contract that normally comes in response to the LP supplying equity capital for investment. Given the GP's opportunity costs of supplying large amounts of capital to investment are likely to be high, the FB discount rate may well exceed the rate required on more patient capital. We will denote that discount rate as $\gamma_i^{FB} \geq r$.

PE fund value in this case, $V_{i,0}^{FB}$, is the sum of equity and debt value, prior to accounting for any potentially distorting effects of GP-LP contracting. Debt value is $D_{i,0}$ as stated in equation (1b). Equity value is a call option on the fund's assets given an exercise price of B_i, drift equal to μ_i, volatility of σ_i, and an equity discount rate of γ_i^{FB}. Denoting FB equity value as $\mathcal{E}_{i,0}^{FB}(B_i)$, we have that

$$\mathcal{E}_{i,0}^{FB}(B_i) = V_{i,0}e^{(\mu_i-\gamma_i^{FB})T_i}N[d_{1,i}] - B_ie^{-\gamma_i^{FB}T_i}N[d_{2,i}] \tag{3}$$

with $d_{1,i}$ and $d_{2,i}$ as defined in (1c), and $V_{i,0}^{FB} = \mathcal{E}_{i,0}^{FB} + D_{i,0}$.

Proposition 1 summarizes the resulting first-best optimal capital structure, in which B_i is chosen to maximize the FB PE fund value, $V_{i,0}^{FB}$.

Proposition 1: When $\gamma_i^{FB} > r$ and $k_i>0$, there exists a unique B_i, B_i^{FB}, that satisfies the FOC: $N[d_{2,t}]\left[e^{-rT_i} - e^{-\gamma_i^{FB}T_i}\right] - e^{-rT_i}n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}} = 0$. B_i^{FB} fully identifies the optimal first-best PE capital structure, with book leverage equal to $\frac{D_{i,0}(B_i^{FB})}{V_{i,0}}$ and market leverage equal to $\frac{D_{i,0}(B_i^{FB})}{V_{i,0}^{FB}}$. Key comparative statics are $\frac{\partial B_i^{FB}}{\partial \gamma_i^{FB}} > 0$; $\frac{\partial B_i^{FB}}{\partial k_i} < 0$; $\frac{\partial B_i^{FB}}{\partial \alpha_i} > 0$.

Proof: See Appendix

Inspection of the FOC in Proposition 1 shows that a modified and stronger version of the choke condition applies. That is, $B_i^{FB} < B_{k_i}^*$ for all B_i, where B_i^{FB} approaches $B_{k_i}^*$ only as γ_i^{FB} becomes large relative to r. First-best leverage is typically well below $B_{k_i}^*$, since, for most reasonable parameter constellations, equity value expressed in (3) decreases rapidly as a function of B_i when $k_i > 0$.

As γ_i^{FB} approaches r from above, optimal leverage goes to zero as the costs of financial distress are no longer traded off against expensive equity. Thus, a high cost of equity capital in which $\gamma_i^{FB} > r$ is necessary for FB leverage to exceed zero. Alternatively, $\gamma_i^{FB} = r$ implies zero leverage. In this case the benefits to positive alpha accrue proportionally to both equity and debt, and in a world with costs of financial distress the FB PE fund capital structure is one with no debt whatsoever.

Figure 2 displays first-best leverage outcomes for base-case parameter values: $r = .02$, $\alpha_i = .05$, $\sigma_i = .30$, $T_i = 7$, $k_i = .40$, $V_{0,i} = 100$, and $\gamma_i^{FB} = .05$. Here we see that $B_i^{FB} = 64.1$ with a corresponding market leverage ratio of .40. These values will serve as useful benchmarks when we examine leveraging incentives when LP investors use leverage to target investment returns.

Figure 2 Here

This analysis enriches our understanding of what it means to have "cheap debt" available to fund PE investment. The relative cost of debt depends on the interactions between three market frictions: costs of financial distress, opportunity cost of equity capital, and alpha. A necessary condition for debt to be "cheap" is that the opportunity cost of equity capital, γ_i^{FB}, is high relative to the cost of debt capital. Alpha by itself cannot generate cheap debt, but does serve to magnify the effects of high equity capital costs in favor of debt. Counteracting these effects are financial distress costs, which all together go into the determination of optimal FB PE fund capital structure.

IV. Incentive Fees and The GP's Leverage Choice Problem

Conditional on the equity capital committed vis-à-vis the PE fund management agreement and on the compensation contract offered by the LP to the GP, the GP simultaneously determines optimal leverage and issues debt to finance the acquisition of the assets to populate the fund. The compensation contract has two possible components: a fixed piece that is paid over the life of the fund by applying a constant percentage to invested capital, and a variable piece that is typically referred to as carried interest. The carried interest component specifies how profits are shared between GP and LP after the "carry return hurdle" is met. The GP's objective is to satisfy its total compensation requirements, which we consider in detail in the next section.

In this section we focus on the variable compensation piece. Recall that GP's in our model are endowed with skill as summarized by α_i, implying no explicit role for effort. This in turn would seem to imply no positive role for a variable compensation contract—the LP could do just as well (or possibly better) by offering a total compensation package with only fixed fees. As noted earlier, the unclear role of incentive fees in PE fund management contracting poses a significant puzzle.

Our proposed solution to the puzzle focuses on satisfying target return objectives of the LP investors. Once the fund manager is chosen by the LP, with tight constraints on the types of assets available for investment, and without variable effort affecting the productivity of assets held in the fund, the GP's only remaining choice variable is leverage. More specifically, carried interest contracting is used as a mechanism that provides the GP incentives to employ a specific amount of fund leverage. The LP does this to satisfy its own target return objectives, which is a central focus of the next section of the paper.

Prior negotiations produce an agreement on total fees. Based on this agreement the LP offers the GP a contract that contains three terms: a fund management fee percentage, a carried interest return hurdle (sometimes referred to as the "preferred return") that must be exceeded for carried interest to be paid, and a carried interest percentage (sometimes referred to as the "promote") that determines the profit split when the carried interest return hurdle is exceeded. The fixed fee component is paid unconditionally, not varying as a function of fund profitability. The variable fee component is the result of an optimization, where, if the GP responds appropriately to incentives contained in the compensation contract, fees as the sum of the fixed and variable contract components equal (in expectation) the pre-negotiated total fee amount. In

this sense, total fees act as a constraint that ensures participation by the GP as fund sponsor, and variable fees function as a constraint that ensures the GP's leverage choice incentives are compatible with LP return objectives. With the fixed asset management fee consuming one degree of contracting freedom, only the other two contract terms—the carried interest return hurdle and the carried interest percentage—are available as instruments align GP-LP incentives.

We denote the $t=0$ value of variable GP fee revenue as $\Phi_i^V(B_i; \psi_i, \rho_i)$, where ψ_i, $\psi_i \geq r$, indicates the carried interest return hurdle and ρ_i, $0 \leq \rho_i \leq 1$, indicates the carried interest percentage. Carried interest, if it is paid, will be based on the time T_i liquidation value of the fund. Specifically, carried interest is not paid unless and until total liquidation value of the fund's assets exceeds the time T_i priority claim payoffs of $B_i + (V_{i,0} - D_{i,0})e^{\psi_i T_i}$, where B_i is balloon debt payment due at time T_i, $V_{i,0} - D_{i,0}$ is equity contributed at the start of the fund's life, and ψ_i is continuously compounded carried interest hurdle rate. Collectively these terms are an exercise price, above which variable compensation is paid and below which the payoff is zero.

Carried interest payments are discounted by the GP at a rate of γ_i, $\gamma_i \geq r$. In our model the GP contributes none of its own equity, instead relying on the LP and the lender for all of the necessary financing.[20] Consequently, financial constraints *per se* do not play an important role in the determination of the GP's discount rate. But, as documented in the PE literature, career concerns as related to, for example, reputation building and the harvesting of current and future income built on reputation, may produce discount rates that vary depending on the GP's age and experience.

With this specification, the GP's optimization problem can be stated as follows:

$$\underset{B_i}{Max}\ \Phi_i^V(B_i; \psi_i, \rho_i) = e^{-\gamma_i T_i}\mathbb{E}_0\left[Max\{0, \rho_i[\tilde{V}_{i,T} - B_i - (V_{i,0} - D_{i,0})e^{\psi_i T_i}]\}\right] \quad (4)$$

Define $\chi_{i,0}(B_i; \psi_i) = B_i + (V_{i,0} - D_{i,0})e^{\psi_i T_i}$, which functions as an exercise price with all terms known with certainty at $t=0$. With this form, equation (4) is recognized as a call option on fund payoffs that exceed those required to pay off priority claims, re-expressed more compactly as,

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[20] It would be straightforward to exogenously account for GP co-investment. Endogenizing co-investment would complicate things without adding much to the analysis.

$$\underset{B_i}{Max}\ \Phi_i^V(B_i;\psi_i,\rho_i) = \rho_i e^{-\gamma_i T_i} \int_{\chi_{i,0}}^{\infty} \left(\tilde{V}_{i,T_i} - \chi_{i,0}\right) f\left(\tilde{V}_{i,T_i}|V_{i,0}\right) d\tilde{V}_{i,T_i} \qquad (4a)$$

Solving the integral in (4a) is routine, where the optimization problem can now be written as,

$$\underset{B_i}{Max}\ \Phi_i^V(B_i;\psi_i,\rho_i) = \rho_i e^{-\gamma_i T_i} \left[V_{i,0} e^{\mu_i T_i} N[h_{1,i}] - \chi_{i,0} N[h_{2,i}]\right] \qquad (4b)$$

$$h_{1,i} = \frac{ln\left[{}^{V_{i,0}}\!/\!_{\chi_{i,0}}\right] + \left(\mu_i + \frac{1}{2}\sigma_i^2\right)T_i}{\sigma_i\sqrt{T_i}},\ h_{2,i} = h_{1,i} - \sigma_i\sqrt{T_i} \qquad (4c)$$

With equations (4b) and (4c) we are now in a position to tackle the optimization problem, where Proposition 2 states the result.

Proposition 2: $\frac{\partial \Phi_i^V}{\partial B_i} = -\rho_i e^{-\gamma_i T_i} N[h_{2,i}]\left[1 - e^{(\psi_i - r)T}\left[N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}}\right]\right] = 0$ **satisfies incentive compatibility. The incentive compatible debt value, B_i^*, exists and is unique for any $k_i \geq 0$. B_i^* does not depend on the carried interest percentage, ρ_i, nor the GP's discount rate, γ_i.**

Proof: See Appendix

Notice that the choke condition, $N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}}$, is positive and decreasing in B_i in the range $B_i \in [0, B_{k_i}^*)$, where, per Lemma 1, $B_{k_i}^*$ satisfies $N[d_{2,i}] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}} = 0$. Since $e^{(\psi_i - r)T} > 1$ except when $\psi_i = r$, $B_i^* < B_{k_i}^*$. When $\psi_i = r$, it is easy to see that $B_i^* = 0$.

Observe that Proposition 2 is fundamentally a separation result, in that both ρ_i and γ_i are irrelevant in the determination of the GP's choice of optimal fund leverage. One implication of separation is that B_i^* is "preference-free," in that it doesn't depend on the GP's discount rate, γ_i. Rather, GP-determined leverage depends only on k_i and α_i as market frictions, with payoff discounting occurring at the benchmark rate, r. This "preference-free" relation may help explain why there is less variation in the carried interest return hurdle across fund managers than one might otherwise expect.

Also observe that an isomorphic mapping exists between the carried interest return hurdle, ψ_i, and the leverage measure, B_i^*, leaving the carried interest percentage, ρ_i as a free parameter. That is, the contract as currently specified is over-identified. This result has important implications for observed PE contracting practices, where it has been noted by various authors that the carried interest percentage seems anchored at 20 percent, with little observed variability around the number (Metrick and Yasuda (2010)). We will take this important issue up again later in the paper.

The incentive compatibility condition reported in Proposition 2 can be restated by isolating the carried interest hurdle value as a function of debt value parameters, including B_i, k_i and α_i. This functional relation can then be used to characterize comparative static relations that exist between B_i^* and ψ_i, as well as B_i^* and other relevant parameter values. The following corollary states the results.

Corollary 1 to Proposition 2: For the relevant range, $B_i^* \in [0, B_{k_i}^*)$, satisfying incentive compatability implies that $\psi_i = r - \frac{1}{T_i} ln \left[N[d_{2,i}] - n(d_{2,i}) \frac{k_i}{\sigma_i \sqrt{T_i}} \right]$. With this, the following comparative static relations obtain: $\frac{\partial B_i^*}{\partial \psi_i} > 0$; $\frac{\partial B_i^*}{\partial k_i} < 0$; $\frac{\partial B_i^*}{\partial \alpha_i} > 0$.

Proof: See Addendix

Inspection of the ψ_i equation reveals that when $B_i^*=0$, $\psi_i=r$, and that ψ_i is increasing in B_i^*. In other words, given a higher carried interest hurdle rate, ψ_i, the GP is incentivized to employ more leverage to maximize variable compensation. It is *never* optimal for the GP to deploy extreme amounts of leverage in response to low or moderate ψ_i's, even when there are no costs of financial distress. This follows because more debt, even when it is cheap debt, makes it more difficult to clear the carried interest hurdle after repaying the debt and delivering a preferred return to the LP.

Figure 3 displays comparative static results, with leverage varying as a function of key parameter values. Leverage is increasing in the carried interest return hurdle, as previously noted.

Higher costs of financial distress discourage leverage, but don't eliminate it, even when losses due to default are extremely high. Increasing alpha reduces the cost of debt to increase leverage.

Figure 3 Here

With this parameter value constellation, although the comparative static in general is indeterminate, increasing asset volatility decreases leverage. This result is not necessarily intuitive in the context of conceptualizing the GP's variable compensation as a call option on the fund's profitability, and follows because increased fund asset volatility, particularly in presence of costs of financial distress, increases debt costs significantly. In other words, the GP bears the cost of downside risk that goes with increased σ_i that is imputed in the cost of debt, which endogenously impacts the exercise price on carried interest payments, more than offsetting the benefits of increasing upside risk. This comparative static result is also interesting in that it illustrates barriers to risk-shifting in fund investment to the extent the lender internalizes these effects into the cost of the debt.

We now compare the GP's optimal leverage choice, B_i^*, to first-best leverage, B_i^{FB}, that obtains with an otherwise identical set of parameter value inputs. The following corollary summarizes the main result.

Corollary 2 to Proposition 2: **For $B_i \in [0, B_{k_i}^*)$ and a given ψ_i and γ_i^{FB}, there exists a unique carried interest hurdle rate, $\widehat{\psi}_i$, $\widehat{\psi}_i > \gamma_i^{FB}$, such that for some $B_i \in [0, B_{k_i}^*)$, B_i simultaneously satisfies FOC's for both first-best leverage (per Proposition 1) and the GP's leverage choice problem (per Proposition 2). As a result, given $\widehat{\psi}_i$, for any $\psi_i > \widehat{\psi}_i$, $B_i^* > B_i^{FB}$ and for any $\psi_i < \widehat{\psi}_i$, $B_i^* < B_i^{FB}$. Lastly, $\frac{\partial B_i^{FB}}{\partial \gamma_i^{FB}}|_{\psi_i = \gamma_i^{FB}} > \frac{\partial B_i^*}{\partial \psi_i}|_{\psi_i = \gamma_i^{FB}}$ for all $\psi_i, \gamma_i^{FB} \geq r$.**

Proof: **See Addendix**

It is always true that for $\psi_i = \gamma_i^{FB}$, $B_i^* < B_i^{FB}$, implying that more leverage is optimal in the first-best case than in the GP's case when it is optimizing leverage to maximize carried interest. This is not necessarily an intuitive result. The relation occurs because of the strong effect that γ_i^{FB}, the opportunity cost of equity capital, has on leveraging incentives in a first-best setting. In

contrast, given the variable contract presented by the LP to the GP, the GP limits leverage because there is a dilution effect as it applies to carried interest. Greater leverage increases the odds of earning carried interest, but contributed equity does not decline proportionally with greater leverage due to the fact that downside risks are internalized into the cost of debt (recall the choke condition per Lemma 1). This decreases GP incentives to lever up relative to first-best. First-best, on the other hand, is not dependent on contributed equity, but rather equity value, which is where the critical difference lies. The comparative static relations stated at the end of Corollary 2 confirm this relation, in which B_i^{FB} increases at a faster rate for increases in the opportunity cost of equity capital than B_i^* increases in response to identified increases in the carried interest return hurdle.

V. Target Returns, Fees and The LP's Contracting Problem

For all of the reasons discussed in the introductory section of the paper, the LP cares only the internal rate of return that a fund can generate based on contributed equity. Once an asset class and fund manager have been singled out for investment consideration, the LP's problem is a relatively simple one: For a given amount of contributed equity at t=0 into fund i, $E_{i,0} = V_{i,0} - D_{i,0}$, and for a targeted return on investment $\lambda_{i,j}$ by LP j, identify and implement leverage that is necessary to meet or exceed the return target.

There are a number of reasons why institutional investors may be interested in using leverage to meet investment return objectives rather than simply moving further out the asset risk curve: 1) Asset classes available for investment are limited at any point in time, and there may be no available fund that offers unlevered expected asset returns high enough to meet the LP return target; 2) The differences in neighboring asset classes in terms of expected returns (α_i's) and asset risk (σ_i's) may be significant, where the LP prefers investing in a lower return asset class and using leverage to meet the return target; 3) There may be regulatory or other constraints that limit the ability of LP to invest in certain higher-risk asset classes, where instead there are fewer constraints on investing in lower risk asset classes that use leverage to boost expected returns; 4) For behavioral or limited rationality reasons, the LP may not be able to accurately distinguish alpha from beta, in the sense that the LP attributes higher expected returns derived from

increases in leverage to real value creation (alpha) rather than pure financial engineering (beta); and 5) The LP may rationally recognize the risk-return implications of leverage, but intentionally obfuscates relations by highlighting investment in lower-risk asset classes while downplaying the leverage used by the fund to achieve higher expected returns.

As we will show, when the costs of financial distress, k_i, are sufficiently large relative to GP fund value creation, α_i, there will be limits to leverage's ability to generate high returns to meet the return target. This in turn may cause the LP, to the extent possible, to move further out the risk curve in search of alternative asset classes that are capable of hitting the targeted investment return.

The fundamental net-of-fee relation considered by the LP is identify a $B_i \geq 0$ that satisfies:

$$e^{\lambda_{i,j}^N T_i} E_{i,0}(B_i) \leq \mathcal{E}_{i,T_i}(B_i) - \Phi_i^V(B_i, \alpha_i) - \Phi_i^F(B_i, \alpha_i) \tag{5}$$

where contributed equity at $t=0$, $E_{i,0}(B_i)$, equals $V_{i,0} - D_{i,0}(B_i)$, $\lambda_{i,j}^N$ is the minimum LP return target, $\mathcal{E}_{i,T_i}(B_i)$ is the time $t=T_i$ expected value of contributed equity prior to the payment of fees, $\Phi_i^V(B_i, \alpha_i)$ denotes the variable fees paid at $t=T_i$ to the GP as carried interest, and $\Phi_i^F(B_i, \alpha_i)$ describes the fixed fees paid by the LP to the GP, brought to the time T_i value. Note the relation in equation (5) is stated as an inequality. This follows since $\lambda_{i,j}^N$, as an IRR, is limited by the RHS value in (5). Furthermore, there may exist B_i's that can increase the levered expected returns to contributed equity above the stated return target, $\lambda_{i,j}^N$, so the return target establishes a minimum expected return on contributed equity.

Prior to the payment of fund management fees, the $t=T_i$ expected value of equity is a call option on the assets of the fund after debt repayment, where

$$\mathcal{E}_{i,T_i}(B_i) = \int_{B_i}^{\infty} [\tilde{V}_{i,T_i} - B_i] f(\tilde{V}_{i,T_i}|V_{i,0}) d\tilde{V}_{i,T_i} = V_{i,0} e^{\mu_i T_i} N[d_{1,i}] - B_i N[d_{2,i}] \tag{6}$$

Note that the $t=T_i$ equity value does not depend on GP preferences.

We will have more to say about the specific characteristics of the fixed asset management fees shortly. For now, it is relevant to keep in mind that variable fees as determined by $\Phi_i^V(B_i, \alpha_i)$ are governed by the relation,

$$\psi_i = r - \frac{1}{T_i} \ln\left[N[d_{2,i}] - n(d_{2,i}) \frac{k_i}{\sigma_i \sqrt{T_i}}\right] \tag{7}$$

as stated in Corollary 1 to Proposition 2. That is, for a given B_i that satisfies equation (5), B_i is uniquely mapped into ψ_i via equation (7) and offered to the GP as a carried interest return hurdle to incentivize the GP to deploy the desired amount of fund leverage.

Gross-of-Fee Return Analysis

Before moving on to assess the effects of alternative fee structures, we can develop some initial understanding by analyzing relations on a gross-of-fee basis in order to see how LP return objectives and leverage interact with one another. Consequently, in this subsection we analyze a truncated version of equation (5) that results in

$$e^{\lambda_{i,j}^G T_i} E_{i,0}(B_i) \leq \mathcal{E}_{i,T_i}(B_i) \tag{5a}$$

where $\lambda_{i,j}^G$ denotes a minimum required gross-of-fee expected return on contributed capital. It will be convenient to isolate the minimum target return by first defining $E_{i,T_i} = e^{rT_i} E_{i,0}$ and writing,

$$\lambda_{i,j}^G = r + \frac{1}{T_i}\left[\ln(\mathcal{E}_{i,T_i}) - \ln(E_{i,T_i})\right] \tag{8}$$

where we now suppress B_i as it determines E_i and $\mathcal{E}_{i,T_i}$.

As a first step in our gross-of-fee return analysis, note that contributed equity, $E_{i,0}$, must be sufficiently small relative to the time $t=T_i$ gross-of-fee expected payoff to equity, $\mathcal{E}_{i,T}$, to ensure a benchmark return of at least r. This immediately eliminates $\alpha_i=0$ asset classes from consideration, as summarized by the following lemma:

<u>Lemma 2</u>: For $\alpha_i=0$, $\lambda_{i,j}^G \leq r$. $\lambda_{i,j}^G = r$ for any $B_i \geq 0$ only when $k_i=0$, and $\lambda_{i,j}^G = r$ only for $B_i=0$ when $k_i > 0$. Otherwise, for $B_i > 0$ when $k_i > 0$, $\lambda_{i,j}^G < r$.

<u>Proof</u>: See Appendix

This result highlights the fact that, when there are no market frictions and no fees, the investor makes the benchmark return of r. This outcome occurs regardless of B_i, which is nothing other than Modigliani-Miller capital structure irrelevance. When financial distress costs are introduced through $k_i > 0$, then the optimal fund capital structure is no debt and an expected return of r. The introduction of expensive debt only serves to decrease the gross-of-fee return below the benchmark return, with $\lambda_{i,j}^G$ decreasing with increases to B_i.

Thus, a necessary condition for $\lambda_{i,j}^G > r$ is $\alpha_i > 0$. Without positive alpha, there is no way to lever the fund into a position in which gross-of-fee expected returns exceed r. The relation between positive alpha and leverage is, however, complicated by the existence of financial distress costs. When $\alpha_i > 0$ and financial distress costs are zero, debt is unambiguously cheap to the LP due to the internalization of positive alpha into the cost of debt. This increases debt proceeds at $t=0$ to decrease contributed equity. Like SWY (2014), in the case of no frictional costs to debt, returns to contributed equity are unbounded as a function of B_i. When $k_i > 0$, the frictional costs of debt will put limits on the return-enhancing ability of "cheap" debt, but only if the costs of financial distress are sufficiently large relative to alpha. The following proposition summarizes the results.

<u>Proposition 3</u>: For $\alpha_i > 0$ and for all $k_i \geq 0$, $\lambda_{i,j}^G = \mu_i$ when $B_i=0$. When costs of financial distress are small relative alpha, such that $k_i < 1 - \frac{V_{i,0}e^{rT_i} - B_i N[d_{2,i}]}{V_{i,0}e^{\mu_i T_i} N[-d_{1,i}]}$ at $B_i = B_{k_i}^*$, $\lambda_{i,j}^G$ increases without bound as B_i approaches $\ddot{B}_i$ from below, where $\ddot{B}_i > 0$ uniquely satisfies $E_{i,T_i}(\ddot{B}_i)=0$. Alternatively, when costs of financial distress are sufficiently large relative to alpha, such that $k_i > 1 - \frac{V_{i,0}e^{rT_i} - B_i N[d_{2,i}]}{V_{i,0}e^{\mu_i T_i} N[-d_{1,i}]}$ at $B_i = B_{k_i}^*$, a unique upper bound for $\lambda_{i,j}^G$ exists in which $\lambda_{i,j}^{G,Max} > \mu_i$, resulting from $B_{i,j}^* > 0$ satisfying the FOC: $N[d_{2,i}]\left[1 - \frac{E_{T_i}}{\varepsilon_{T_i}}\right] - n(d_{2,i})\frac{k_i}{\sigma_i\sqrt{T_i}} = 0$.

Proof: See Appendix

Figure 4 shows on gross-of-fee investment IRRs vary as a function of B_i. Panel A is the base case. In this case, $k_i < 1 - \frac{V_{i,0}e^{rT_i} - B_i N[d_{2,i}]}{V_{i,0}e^{\mu_i T_i} N[-d_{1,i}]} = .29$ at $B_i = B^*_{k_i}$, which is the value that determines whether $\lambda^G_{i,j}$ is bounded or not. At $B_i=0$, $\lambda^G_{i,j} = \mu_i = .07$, peaking at $\lambda^{G,Max}_{i,j} = .120$ with the corresponding $B^*_{i,j} = 153.5$. The market leverage ratio in this case is .89, which compares to first-best leverage of .40 using the same parameter value constellation. If the LP investor employed 40 percent leverage to finance fund investment, the gross-of-fee return would be about 10 percent instead of 12 percent, which is eight rather than 10 percent in excess of the baseline return of 2 percent.

Figure 4 Here

Panel B, C, D, and E of Figure 4 display how $\lambda^G_{i,j}$ varies as a function of B_i for alternative parameter values. In Panel B, $\alpha_i = .02, .10$ and $.15$ in addition to the base case value of .05. In the cases of $\alpha_i = .10$ and $.15$, the target IRR function blows up with infinite returns realized at $\ddot{B}_i = 171.8$ and $\ddot{B}_i = 133.3$, respectively. In the case of $\alpha_i = .02$, leverage at the maximum is much more modest than it is at the maximum when $\alpha_i = .05$, in which now $B^*_{i,j} = 48.3$ rather than 153.5. Panel C shows outcomes for alternative k_i's, where we recall that the function blows up for $k_i = 0, .20$. We note that when costs of financial distress increase from the base case k_i value of .40 to 1.0, leverage reduces substantially to $B^*_{i,j} = 55.25$ at the value of B_i that maximizes the target IRR.

Panels D and E display relations for alternative fund-debt maturity dates and asset volatilities, respectively. Longer fund-debt maturities correspond with unbounded $\lambda^G_{i,j}$'s, but we note that extremely high leverage is required to cause much separation in target IRR's as the fund-debt maturity varies between 3 and 15 years. The results for changing asset volatilities are quite interesting and rather unintuitive. These simulations show that low asset volatility funds dominate high volatility funds, in the sense that the low volatility funds achieve higher peak IRR's at lower leverage levels than high volatility funds. A fund maturity of $T_i = 7$ years

~ 27 ~

combined with relatively high costs of financial distress with $k_i=.40$ imply that asset volatility creates a significant drag on levered fund returns.

Total Fees and Net-of-Fee Return Analysis

We now move on to consider LP returns on a net-of-fee basis. Total fees result from a negotiation between the GP and LP that establishes a participation constraint that must be met by the LP. An upper bound on fees will, in general, be a function of the GP's endowed alpha when that alpha is sufficiently positive. This upper bound on fees is analogous to SWY's (2014) notion of a breakeven alpha, where instead of taking fees as given and calculating a minimum alpha necessary to cover the fees, we come the other direction by taking alpha as given and determine total fees that generate net-of-fee returns that equal or exceed the baseline return, r.

Throughout our analysis in this section we require $\alpha_i > r$. It will also be useful to isolate the minimum required net-of-fee return target, $\lambda_{i,j}^N$, as originally stated in equation (5), as a function of contributed equity, terminal gross-of-fee equity value and total fees:

$$\lambda_{i,j}^N \leq r + \frac{1}{T_i} \left[ln \left[\mathcal{E}_{i,T_i} - \Phi_i^V(B_i, \alpha_i) - \Phi_i^F(B_i, \alpha_i) \right] - ln \left(E_{i,T_i} \right) \right] \tag{5'}$$

where the focus is on determining $\lambda_{i,j}^N$ as a function of B_i. By comparing equation (5') with (5a), it is immediately apparent that, for any given B_i, fees reduce the initial gross-of-fee difference between $\mathcal{E}_{i,T_i}$ and $E_{i,0}$ to decrease $\lambda_{i,j}^N$ relative to $\lambda_{i,j}^G$. In particular, the downward shift in $\lambda_{i,j}^N$ relative to $\lambda_{i,j}^G$ may limit the range of B_i's capable of producing targeted returns that exceed the baseline return, r.

Variable fees are set by the incentive compensation contract that was analyzed in detail in section IV, with variable fees paid at time T_i according to

$$\Phi_i^V(B_i, \alpha_i) = \rho_i \left[V_{i,0} e^{\mu_i T_i} N[h_{1,i}] - \chi_{i,0} N[h_{2,i}] \right] \tag{9}$$

where we recall that ρ_i denotes the carried interest percentage paid to the GP when the carried interest hurdle return is met, and $\psi_i \geq r$ is the endogenously determined carried interest hurdle

return that is implicit in the exercise price, $\chi_{i,0}$. Note that variable fees are independent of GP preferences. Also recall that leverage as expressed by B_i is implicit in $\chi_{i,0}$, $N[h_{1,i}]$, and $N[h_{2,i}]$.

Fixed fees are paid continuously over the life of the fund. These fees are not contingent on fund performance. Let ϕ^F denote the asset management fee percentage that is applied to contributed capital, $E_{i,0}$. In general, this fee may depend on both B_i and α_i. Total fixed fees valued as of time T_i are thus,

$$\Phi_i^F(B_i, \alpha_i) = \int_0^{T_i} \phi^F E_{i,0} e^{\gamma_i(T_i - t)}\, dt = \phi^F E_{i,0} \left[\frac{e^{\gamma_i T_i} - 1}{\gamma_i}\right] \qquad (10)$$

Total fees are the sum of variable and fixed fees. We note that total fees will in general affect the LP's leverage choice. This follows because fees affect net-of-fee returns, which then affect the LP's leverage choice in the context of meeting net-of-fee target return objectives. For example, if fixed fees are set exogenously according to standard industry practice, such a practice will encourage further fund levering on the part of the LP in order to dilute the effect of these fees based on the total value of assets contained in the fund. Thus, fees not only create a drag on net-of-fee returns, they also generally affect the shape of the LP's levered net-of-fee return distribution as expressed in equation (5').

We will now consider three fee regimes that illustrate both the realities of PE fee structures as well as various aspects of how fees can be used to address the respective compensation and investment return objectives of the GP and LP. Throughout we maintain the assumption that the GP is focused on satisfying total fee requirements, with indifference between the split of fixed and variable fees. The third fee structure regime we consider is one where we endogenize the effects of alpha to determine a break-even upper bound on total fees the GP can charge the LP.

Fee Regime #1: Thus far we have described a central role for the carried interest return hurdle rate, but not for the other contracting variables of the carried interest percentage and the asset management fee. In this case we follow the literature and, with the exception of the carried interest hurdle rate, ψ_i, set the compensation contract parameters exogenously at values that match those observed in practice. In particular, we will fix the asset management fee percentage

to $\phi_i^F = \bar{\phi}_i^{F1}$ and the GP's carried interest percentage to $\rho_i = \bar{\rho}_i$, where $\bar{\phi}_i^{F1}$ and $\bar{\rho}_i$ are set based on observed industry practice. In this case,

$$\Phi_i^{V1} = \bar{\rho}_i \left[V_{i,0} e^{\mu_i T_i} N[h_{1,i}] - \chi_{i,0} N[h_{2,i}] \right] \tag{11a}$$

and

$$\Phi_i^{F1} = \bar{\phi}_i^{F1} E_{i,0} \left[\frac{e^{\gamma_i T_i} - 1}{\gamma_i} \right] \tag{11b}$$

Figure 5 displays base case net-of-fee as well as gross-of-fee returns when $\bar{\phi}_i^{F1} = .02$ and $\rho_i = .20$, as is often observed in practice. Over the relevant range for which $\lambda_{i,j}^N$ is increasing, there is about a three percent annual differential between gross-of-fee and net-of-fee returns. At the maximum, $\lambda_{i,j}^{N-Max} \approx .09$, which is a 7.0 percent return in excess of the benchmark rate. To achieve this return, net-of-fee leverage increases somewhat from the corresponding gross-of-fee leverage, going from $B_{i,j}^* = 153.5$ to $B_{i,j}^* = 164.7$. This outcome occurs because, as previously noted, scaling up total fund size through leverage creates a marginal decrease in fixed fee parameters that do not explicitly reference contributed equity.

Figure 5 Here

Fees in this base case are 34.6 percent of contributed equity, with approximately one-half of the fees allocated to the fixed asset management fee and the other one-half allocated to carried interest. If the targeted net-of-fee return is 5.0 percent in excess of the benchmark rate, the corresponding $B_{i,j} \approx 70$. The resulting incentive compatible carried interest hurdle, ψ_i equals about 9.5 percent, which is in the range of 6 to 10 percent that is often observed. If, however, the LP wants to target the maximum return of 10.0 percent in excess of the benchmark rate, the incentive compatible carried interest hurdle, ψ_i, jumps to increase to about 28.5 percent. These results compare to a first-best leverage of $B_i^{FB} = 64.1$, which results in ψ_i equal to 8.5 percent.

Fee Regime #2: To illustrate how the carried interest percentage can be endogenized and used as a contract parameter to meet the GP's participation constraint, in this case we fix the asset

management fee by letting $\phi_i^F = \bar{\phi}_i^{F2}$. With only two contract parameters to be determined, the carried interest percentage, ρ_i, now assumes a central role by satisfying the GP's participation constraint in which total fees equal a certain percentage of contributed LP capital. Denoting this percentage as $\bar{\pi}_i$, participation is constrained by the relation,

$$\Phi_i^{V2} \geq Max\left\{0, \bar{\pi}_i E_{i,0} - \bar{\phi}_i^{F2} E_{i,0}\left[\frac{e^{\gamma_i T_i}-1}{\gamma_i}\right]\right\} \tag{12}$$

where the time T_i value of the asset management fee component is $\Phi_i^{F2} = \bar{\phi}_i^{F2} E_{i,0}\left[\frac{e^{\gamma_i T_i}-1}{\gamma_i}\right]$ per

equation (11b), and where $\bar{\pi}_i E_{i,0} - \bar{\phi}_i^{F2} E_{i,0}\left[\frac{e^{\gamma_i T_i}-1}{\gamma_i}\right]$ can be rewritten as $E_{i,0}\left[\bar{\pi}_i - \bar{\phi}_i^{F2}\left[\frac{e^{\gamma_i T_i}-1}{\gamma_i}\right]\right]$.

The LP's interest is to minimize total fees, implying that we can treat the participation constraint in (12) as an equality. As a consequence, the optimal carried interest percentage, ρ_i^*, is determined as follows:

$$\rho_i^* = \Phi_i^{V2}\left[V_{i,0}e^{\mu T}N[h_{1,i}] - \chi_{i,0}N[h_{2,i}]\right]^{-1} \tag{13}$$

While Φ_i^{V2} is constant as a function of fund leverage, the terms inside the bracket will vary as a function of B_i. Based on results reported in Lemma 1 and Proposition 2, including the fact that the carried interest percentage is determined separately from the carried interest hurdle rate, the optimal carried interest percentage is unambiguously increasing in fund leverage, B_i.

Figure 6 displays $\lambda_{i,j}^N$ as a function of B_i, where total fees are a constant percentage of committed capital. For this figure we set total fees equal to 40 percent of contributed equity, with $\bar{\phi}_i^{F1} = .02$ as in the prior case. Then ρ_i^* as defined in equation (13) is determined to ensure the 40 percent total fee requirement is satisfied as a participation constraint. In this case the fee drag approaches 4 percent per year over the relevant range of B_i. To achieve the 40 percent total fee requirement, $\rho_i^* = .246$, which slightly exceeds the 20 percent carried interest value often observed in practice. Also note that in this case, with total fees as a constant percentage of contributed equity, $\lambda_{i,j}^{G-Max}$ and $\lambda_{i,j}^{N-Max}$ are realized at the same $B_{i,j}^* = 153.5$. This demonstrates that incentives to increase leverage net-of-fees depends on the fee regime under consideration.

Figure 6 Here

Fee Regime #3: In this case the GP determines total fees based in part on the alpha it generates to the benefit of fund investors. In the extreme case, the GP extracts all value created for the LP by setting total fees so that $\lambda_{i,j}^N \leq r$, with $\lambda_{i,j}^N = r$ if $\lambda_{i,j}^N > r$ after extracting a minimum total fee amount that is required for participation. Full-fee extraction thus implies that fees are set so that the LP never does better than the breakeven rate r on a net-of-fee basis, regardless of the B_i chosen by the LP. Otherwise, if $\lambda_{i,j}^N \leq r$ for any B_i after extracting the minimum fee amount, the GP settles for the minimum fee (assuming the LP is willing to enter into the contract to begin with).

A full-fee extraction strategy will as a result make the LP indifferent regarding fund leverage choices, implying that, although potentially lucrative for the GP, the strategy is self-defeating as it not only eliminates excess net-of-fee returns but also neuters the LP's ability to target investment returns through financial engineering. This causes us to consider a total fee setting approach in which fees are either, i) the minimum total fee amount when $\lambda_{i,j}^N \leq r$ for any B_i after the minimum fee is paid, or ii) the weighted average of the minimum fee and the full-fee extraction amount when $\lambda_{i,j}^N > r$ for some range of B_i's after the minimum fee is subtracted from fund profits.

We formalize our approach as follows. The minimum total fee is set as analyzed previously under fee regime #2. It is a combination of fixed and variable fees that sum to a fixed percentage of contributed capital that is required by the GP for participation. Denote the minimum total fee in this case as $\breve{\Phi}_i^{T3} = \bar{\pi}_i E_{i,0}$. We will now allow for two methods of implementing the minimum total fee, which is to either set fixed fees in advance and vary the carried interest percentage, ρ_i, as we did in the previous fee regime case, or fix ρ_i in advance and vary the fixed fee percentage, ϕ_i^F, to satisfy the participation constraint.[21]

When $\lambda_{i,j}^N > r + \frac{1}{T_i}\left[ln\left[\mathcal{E}_{i,T_i} - \breve{\Phi}_i^{T3}\right] - ln\left(E_{i,T_i}\right)\right]$, the GP will be in a position to charge a weighted average fee. The fully extracted fee amount, which we denote as $\widehat{\Phi}_i^{T3}(B_i, \alpha_i)$, is

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[21] A third method of course would be to vary both ρ_i and ϕ_i^F to satisfy the minimum fee requirement.

calculated to result in $\lambda_{i,j}^N = r$. This breakeven fee calculation is implemented by taking equation (5) as an equality, letting $\lambda_{i,j}^N = r$, and subsequently finding $\widehat{\Phi}_i^{T3}(B_i, \alpha_i)$ to satisfy,

$$\widehat{\Phi}_i^{T3}(B_i, \alpha_i) = \mathcal{E}_{i,T_i}(B_i) - E_{i,T_i}(B_i) \tag{14}$$

In other words, the maximum fee, whose value is brought forward to time T_i, is the difference between the time T_i gross-of-fee expected equity value and the time T_i value of contributed equity compounded at the benchmark rate, r.

Applying the definitions of $\mathcal{E}_{i,T_i}(B_i)$ and $E_{i,T_i}(B_i)$ and simplifying, (14) can be rewritten as,

$$\widehat{\Phi}_i^{T3}(B_i, \alpha_i) = V_{i,0} e^{rT_i}\left[e^{\alpha_i T_i}\left(1 - kN[-d_{1,i}]\right) - 1\right] \tag{14a}$$

With this formulation, a net-of-fee break-even α_i can be calculated that identifies whether, for any given B_i, $\widehat{\Phi}_i^{T3}(B_i, \alpha_i) \gtrless \breve{\Phi}_i^{T3}$. That is, we can use (14a) to identify a break-even $\hat{\alpha}_i$, $\hat{\alpha}_i > r$, such that for any $\alpha_i > \hat{\alpha}_i(B_i)$, $\widehat{\Phi}_i^{T3}(B_i, \hat{\alpha}_i) > \breve{\Phi}_i^{T3}$. Finding $\hat{\alpha}_i(B_i)$ amounts to substituting $\breve{\Phi}_i^{T3}$ for $\widehat{\Phi}_i^{T3}$ in (14a) and solving for $\hat{\alpha}_i$. After doing so the following relation falls out:

$$\hat{\alpha}_i(B_i) = \frac{1}{T_i}\left[ln\left[1 + \frac{e^{-rT_i}\breve{\Phi}_i^{T3}}{V_{i,0}}\right] - ln\left[1 - k_i N[-d_{1,i}]\right]\right] \tag{15}$$

where $\hat{\alpha}_i$ is implicit in $d_{1,i}$.[22]

Lastly, assuming $\alpha_i > \hat{\alpha}_i(B_i)$ for some range of B_i's, in this range total fees are calculated as a weighted average of the minimum fee and the full-fee extraction amount, expressed as follows:

$$\Phi_i^{T3}(B_i, \alpha_i) = \eta\widehat{\Phi}_i^{T3}(B_i, \alpha_i) + (1 - \eta)\breve{\Phi}_i^{T3}(B_i, \alpha_i) \tag{16}$$

where η denotes the weighting factor that may depend on the bargaining power of the GP relative to the LP. Once the total fee is determined, then, as discussed previously, the split between fixed and variable fees can be done in order to identify the compensation contract variables ϕ_i^F and ρ_i.

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[22] Alternatively, we could have explicitly isolated a break-even k_i, below which $\widehat{\Phi}_i^{T3}$ exceeds $\breve{\Phi}_i^{T3}$.

Figure 7 displays the results where we impose a minimum fee, $\breve{\Phi}_i^{T3}$, equal to 40 percent of contributed equity, and where the weight between the minimum fee and full-extraction fee is $\eta = .50$. Here the effects of full-fee extraction are pronounced, with a fee drag of between 4.5 to in excess of 6.0 percent in the relevant range. For this case, as with fee regime #2, we fix the asset management fee at .02 and solve for ρ_i^* as defined in equation (13) to ensure the total fee requirement is satisfied as a participation constraint. As a result of the effects of full-fee extraction, ρ_i^* increases to in excess of 60 percent in order to meet the high total fee payment exceeds 50 percent of contributed equity. Also note that as in the previous fee regime, $\lambda_{i,j}^{G-Max}$ and $\lambda_{i,j}^{N-Max}$ are realized at the same $B_{i,j}^* = 153.5$. This occurs because the full-fee extraction component to total fees is leverage neutral, while the minimum fee generates leverage equality, on a gross- versus net-of-fee basis, implying that a weighting of the two components generates the identical leverage result documented in regime #2.

Figure 7 Here

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Appendix

For all of the proofs, we will suppress subscripts wherever doing so does not introduce any ambiguity into the meaning of the variables.

Proof of Lemma 1:

Starting with equations (1b) and (1c), after doing a small bit of algebra and using the known formula for the standard normal pdf, we have that $\frac{\partial D_0}{\partial B} = e^{-rT}\left[N[d_2] - \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{1}{2}d_2^2}\right] + (1 - k)V_0e^{\alpha T}\left[\frac{1}{B\sigma\sqrt{T}\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\right]$. From (1c), $d_1^2 = d_2^2 + 2d_2\sigma\sqrt{T} + \sigma^2 T$. Substituting this into the prior equation, using the definition of d_2^2 from (1c), and after completing the squares in the exponents and doing some algebra, we end up with $\frac{\partial D_0}{\partial B} = e^{-rT}\left[N[d_2] - \frac{1}{\sigma\sqrt{T}}n(d_2)\right] + (1 - k)e^{-rT}\frac{1}{\sigma\sqrt{T}}n(d_2) = e^{-rT}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$.

To prove the existence and uniqueness of a finite B_k^* that satisfies $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$, we first note that $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is everywhere continuous, $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) = 1$ for $B=0$ and that $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) \to 0$ as $B \to \infty$. Now, it is true that $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} \to 0$ from below (i.e., the quantity is negative for B large), since, for this to be true, $\frac{k}{\sigma\sqrt{T}} > \frac{N[d_2]}{n(d_2)}$ for any $k>0$ as B gets large. Applying L'Hospital's rule to the RHS of the inequality shows that it goes to zero in the limit, confirming that $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) \to 0$ from below. Next, take the derivative of $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2)$ with respect to B, which results in $n(d_2)\frac{\partial d_2}{\partial B}\left[1 + d_2\frac{k}{\sigma\sqrt{T}}\right]$. The terms outside the bracket together are negative. The term inside the bracket is initially positive for B small and then eventually turns negative for some B sufficiently large. This implies that the slope of $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is initially negative as a function of B, but then turns positive for some unique B sufficiently large, and then stays positive thereafter. This is all that is needed for existence and uniqueness of B_k^*, since, for the above collection of facts to be true, it must be the case that there is a single crossing in which there is one and one B for which $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$. Finally, given these facts, in the range of $B \in [0, B_k^*)$ it immediately follows that $\frac{\partial D_0}{\partial B} > 0$ and $\frac{\partial^2 D_0}{\partial B^2} < 0$ when $k>0$. QED

Proof of Proposition 1:

From equations (1b) and (3), and using $V_0^{FB} = \mathcal{E}_0^{FB} + D_0$, we recognize $\mathcal{E}_0^{FB}$ as a call option with exercise price B. Using standard comparative static results for European call options and the comparative static for D_0, we have that, $\frac{\partial V_0^{FB}}{\partial B} = N[d_2]\left[e^{-rT} - e^{-\gamma^{FB}T}\right] - e^{-rT}n(d_2)\frac{k}{\sigma\sqrt{T}}$. Following similar logic to that employed to prove lemma 1, this comparative static has exactly the same properties as $\frac{\partial D_0}{\partial B}$ with respect to B uniquely satisfying the FOC, $\frac{\partial V_0^{FB}}{\partial B} = 0$. In particular, there is a single crossing of $\frac{\partial V_0^{FB}}{\partial B}$ at zero when $B = B^{FB}$ and $\frac{\partial V_0^{FB}}{\partial B}$ has a negative derivative with respect to B at $B = B^{FB}$, implying that $\left[e^{-rT} - e^{-\gamma^{FB}T}\right] + d_2 e^{-rT}\frac{k}{\sigma\sqrt{T}} > 0$ at $B = B^{FB}$. This proves existence and uniqueness of B^{FB}, and hence an interal optimum capital structure. Inspection of $\frac{\partial V_0^{FB}}{\partial B}$ reveals that $\frac{\partial B^{FB}}{\partial \gamma^{FB}} > 0$ and $\frac{\partial B^{FB}}{\partial k} < 0$. To prove $\frac{\partial B^{FB}}{\partial \alpha} > 0$ is a bit more involved. By taking the derivative of $\frac{\partial V_0^{FB}}{\partial B}$ with respect to α at $B = B^{FB}$ we obtain $n[d_2]\frac{\partial d_2}{\partial \alpha}\left[\left[e^{-rT} - e^{-\gamma^{FB}T}\right] + d_2 e^{-rT}\frac{k}{\sigma\sqrt{T}}\right]$. The two terms outside the brackets are positive, as is the term inside the brackets (for the reasons discussed previously), implying that $\frac{\partial B^{FB}}{\partial \alpha} > 0$. QED

Proof of Proposition 2:

From equations (4b) and (4c) the FOC is: $\frac{\partial \Phi_V}{\partial B} = \rho e^{-\gamma T}\left[V_0 e^{\mu T}n(h_1)\frac{\partial h_1}{\partial B} - N[h_2]\frac{\partial \chi_0}{\partial B} - \chi_0 n(h_2)\frac{\partial h_2}{\partial B}\right] = 0$. Recalling that $\chi_0 = B + (V_0 - D_0)e^{\psi T}$, it follows that $\frac{\partial \chi_0}{\partial B} = 1 - e^{(\psi - r)T}\left[N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2)\right]$, $\psi, \gamma \geq r$. Subbing this into the FOC and utilizing well known comparative static relations for call options with respect to B, the FOC simplifies to $\frac{\partial \Phi_V}{\partial B} = -\rho e^{-\gamma T}N[h_2]\left[1 - e^{(\psi - r)T}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]\right] = 0$. For $\rho > 0$ the FOC comes down to equating the terms inside the brackets to zero. For existence and uniqueness, the same logic spelled out in the proof to lemma 1 apply. The lack of dependence on ρ and γ is based on inspection of the FOC above. QED

Proof of Corollary 1 to Proposition 2:

$\psi = r - \frac{1}{T}ln\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$ follows directly from the final form of the FOC expressed in the proof to proposition 2. The comparative static $\frac{\partial B^*}{\partial \psi} > 0$ follows from $\frac{\partial \psi}{\partial B} > 0$ in the above functional relation, since $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) > 0$ and the derivative of this quantity is negative, per the proof of lemma 1. Implicit differentiation generates the other two stated comparative static

relations, where $\frac{\partial B^*}{\partial k} < 0$ follows from the fact that $\frac{-\partial \psi}{\partial k} < 0$ and $\frac{\partial B^*}{\partial \alpha} > 0$ follows from the fact that $\frac{-\partial \psi}{\partial \alpha} > 0$. QED

Proof of Corollary 2 to Proposition 2:

The FB FOC in the leverage choice problem can be expressed as: $e^{-rT}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right] = N[d_2]e^{-\gamma^{FB}T}$. Similarly the FOC applying to the GP in its leverage choice problem can be expressed as $e^{-rT}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right] = e^{-\psi T}$. Also recall that the range for both γ^{FB} and ψ is $[r,\infty)$. Inspection of these FOC's reveals that when $\gamma^{FB} > \psi$, $N[d_2]e^{-\gamma^{FB}T} < e^{-\psi T}$, implying that $B^{FB} > B^*$ due to the fact that the choke condition (bracketed term on the LHS of each FOC) is decreasing in B. Now consider the case of $\gamma^{FB} < \psi$. In this case pick any $B \in [0, B_k^*)$ and plug it into both FOC's. For the given B, there will exist a $\hat{\psi}_B$ such that $N[d_2]e^{-\gamma^{FB}T} = e^{-\hat{\psi}_B T}$. Now it is clear that, for $\psi > \hat{\psi}_B$, $B^{FB} < B^*$, for $\gamma^{FB} < \psi < \hat{\psi}_B$, $B^{FB} > B^*$, and for $\psi = \hat{\psi}_B$, $B^{FB} = B^*$.

As for the comparative static relations, rewrite the FOC's as $\psi = r - \frac{1}{T}ln\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$ and $\gamma^{FB} = r - \frac{1}{T}\left[ln\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right] - ln[N[d_2]]\right]$. Given that the choke condition is strictly decreasing in B, inspection of these two FOC relations reveals that both equations are increasing in B and that the first FOC is increasing at a faster rate than the second FOC. The comparative statics we are interested in are the inverse of these stated relations, implying that both comparative statics are positive and that $\frac{\partial B_i^{FB}}{\partial \gamma_i^{FB}}\big|_{\psi_i = \gamma_i^{FB}} > \frac{\partial B_i^*}{\partial \psi_i}\big|_{\psi_i = \gamma_i^{FB}}$. QED

Proof of Lemma 2:

Let $\alpha = 0$. Then $E_T = V_0 e^{rT} - BN[d_2] - (1-k)V_0 e^{rT}N[-d_1]$ and $\mathcal{E}_T = V_0 e^{rT}N[d_1] - BN[d_2]$. For $k=0$, $E_T = \mathcal{E}_T$, implying that $\lambda^G = r$ per equation (8). When $k>0$, $E_T \geq \mathcal{E}_T$, implying $\lambda^G \leq r$, where simple inspection reveals that $E_T = \mathcal{E}_T$ only when $B=0$. QED

Proof of Proposition 3:

Here $\alpha > 0$. There are several important facts that are central to the proof: 1) $\mathcal{E}_T > 0$ for all $B \in [0, B_k^*)$, which is the relevant range of B's per the choke condition in lemma 1; 2) $E_T \lessgtr 0$ at $B = B_k^*$; 3) $E_T < \mathcal{E}_T$ at $B=0$; 4) Both E_T and $\mathcal{E}_T$ are decreasing in B, with $\mathcal{E}_T$ decreasing at a faster rate than E_T in the relevant range, $B \in [0, B_k^*)$. The second fact in which it is possible for $E_T < 0$ at $B = B_k^*$ follows from the fact that assets are purchased at their pre-alpha value, V_0, whereas debt issuance proceeds incorporate alpha. Thus when alpha is high relative to costs of financial distress, k, in combination with fact 4, for certain parameter value constellations $E_T = 0$ for

leverage $B < B_k^*$. Fact 4 comes directly from comparative statics for E_T and $\mathcal{E}_T$, where $\frac{\partial \mathcal{E}_T}{\partial B} = -N[d_2]$ and $\frac{\partial E_T}{\partial B} = -N[d_2] + n(d_2)\frac{k}{\sigma\sqrt{T}}$. The choke condition implies the latter comparative static is negative in the relevant range.

To start, inspection of equation (8) at $B=0$ immediately reveals that $\lambda^G = \mu$. Now consider two cases. In the first case, suppose that $E_T < 0$ at $B = B_k^*$. From equation, in this case $\lambda^G \to \infty$ as E_T approaches zero for some $B < B_k^*$. $E_T < 0$ at $B = B_k^*$ implies that $E_T = V_0 e^{rT} - BN[d_2] - (1-k)V_0 e^{\mu T}N[-d_1] < 0$ at $B = B_k^*$, which in turn implies $k < 1 - \frac{V_0 e^{rT} - BN[d_2]}{V_0 e^{\mu T}N[-d_1]}$ at $B = B_k^*$. Because E_T is strictly decreasing in B, there exists a unique $\ddot{B} < B_k^*$ such that $E_T = 0$ at $B = \ddot{B}$, implying that λ^G increases without bound at that point.

Now consider the second case in which $E_T > 0$ at $B = B_k^*$. Here λ^G does not increase without bound for $B \in [0, B_k^*)$. Consequently, we want to consider the existence and uniqueness of a maximum $\hat{\lambda}^G$ at some $B < B_k^*$. Taking the FOC with respect to B for equation (8), and after doing some algebra, we obtain: $\frac{\partial \lambda^G}{\partial B} = \frac{1}{T}\left[\frac{1}{\mathcal{E}_T}\frac{\partial \mathcal{E}_T}{\partial B} - \frac{1}{E_T}\frac{\partial E_T}{\partial B}\right] = 0$, which becomes, $N[d_2]\left[1 - \frac{E_T}{\mathcal{E}_T}\right] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$. The form of the FOC in comparison to the choke condition of $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ and the fact that both E_T and $\mathcal{E}_T$ are positive at $B = B_k^*$ imply the existence at least one $\ddot{B} < B_k^*$ that satisfies the FOC. Uniqueness of $\ddot{B} < B_k^*$ is ensured due to $\frac{\partial \mathcal{E}_T}{\partial B} < \frac{\partial E_T}{\partial B}$, implying that $\left[1 - \frac{E_T}{\mathcal{E}_T}\right]$ is decreasing in B, and because $N[d_2]$ decreases as a function of B for all $B \in [0, B_k^*)$.

The uniqueness of the solution to the FOC along with the fact that $\frac{\partial \lambda^G}{\partial B} > 0$ at $B=0$ implies the maximum exceeds $\lambda^G = \mu$. QED
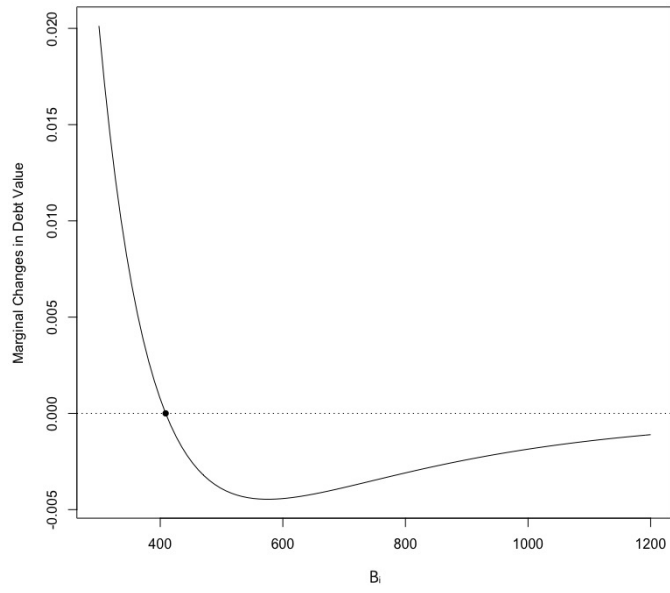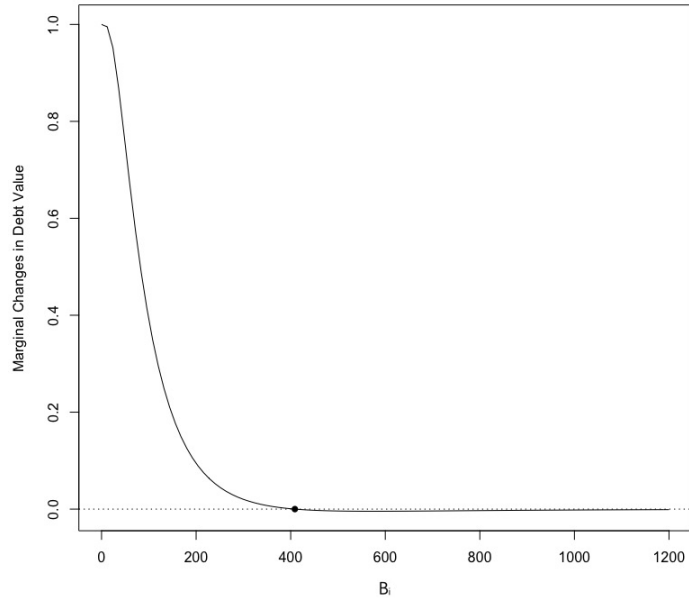
Figure 1

The Choke Condition
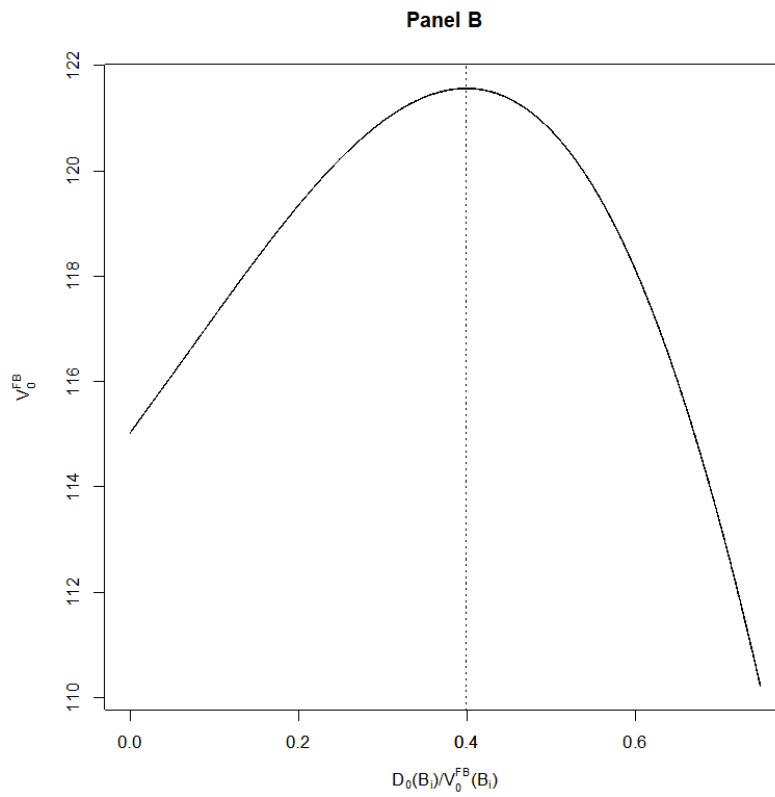
Figure 2

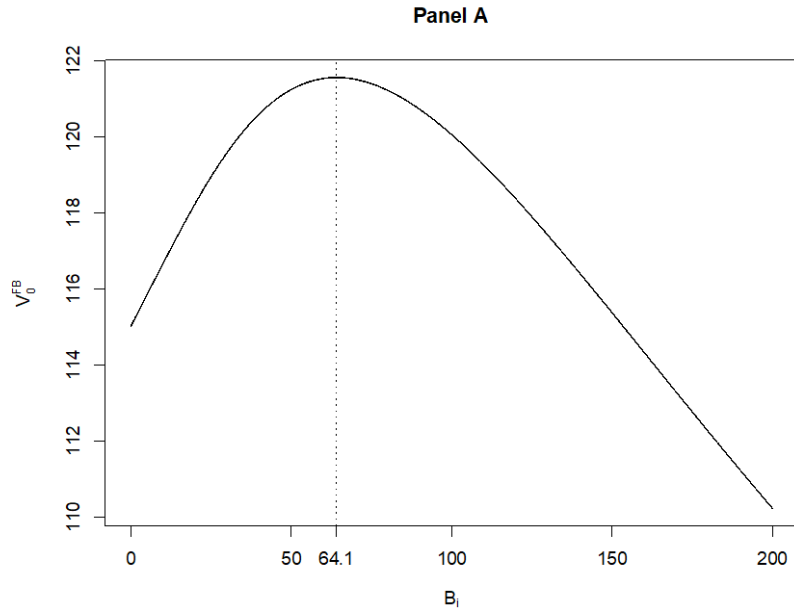## First-Best Capital Structure

Panel A

Panel B

Figure 3

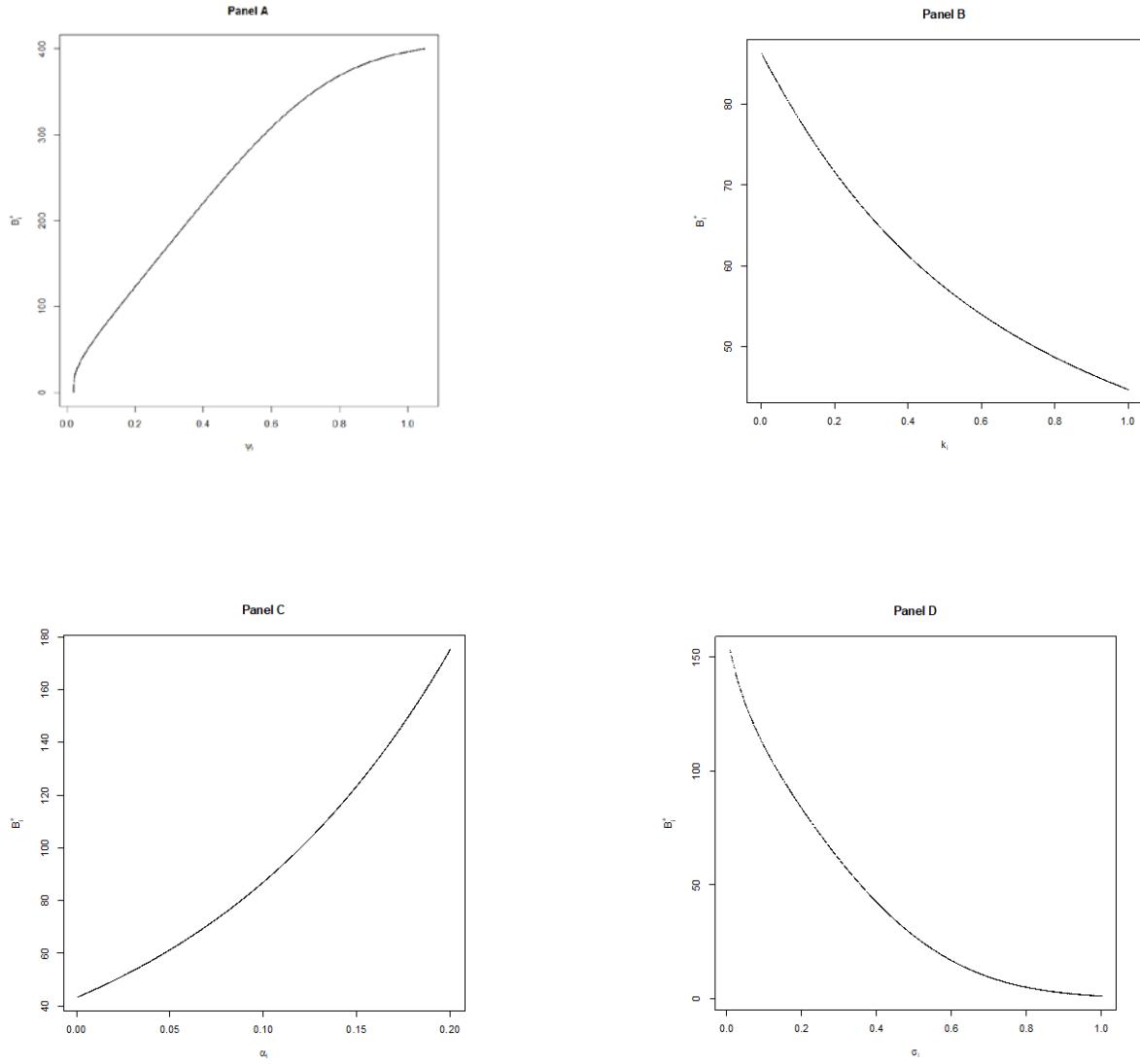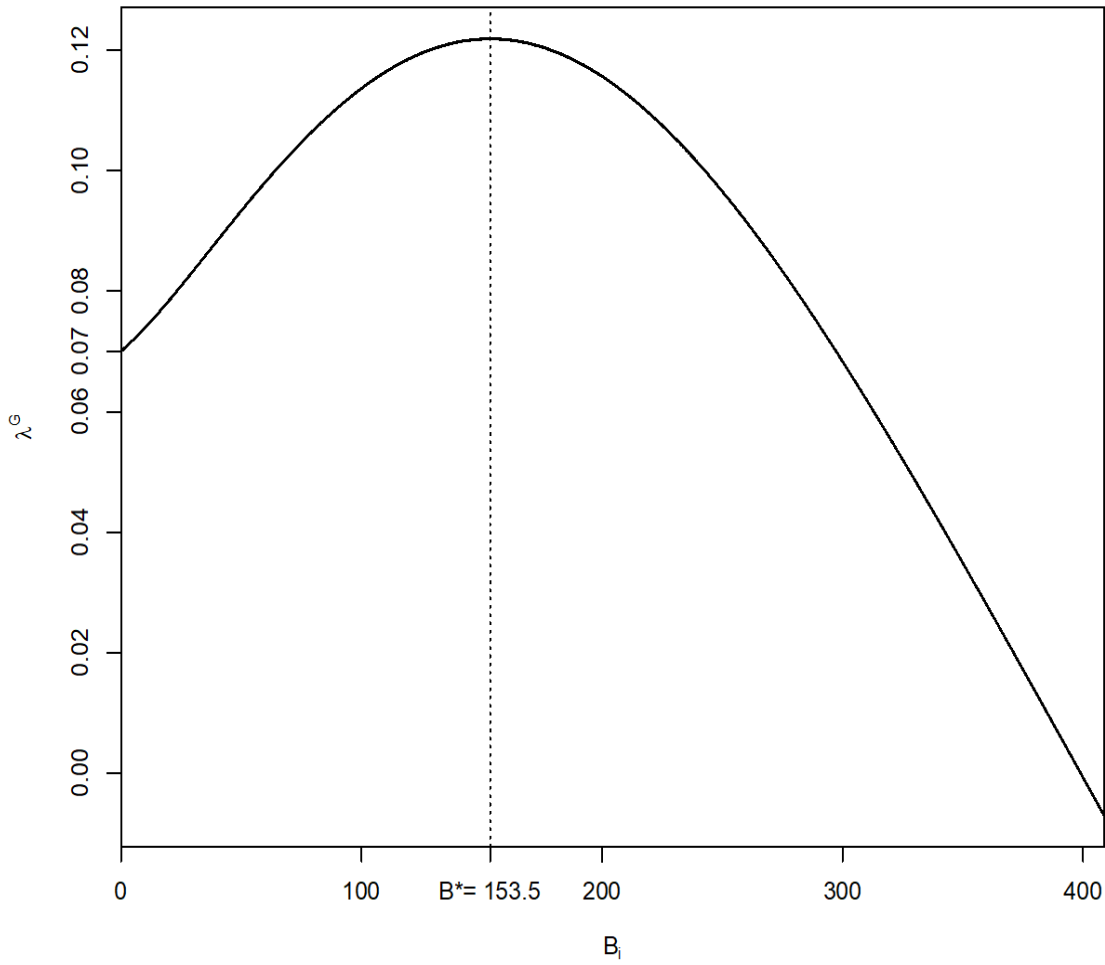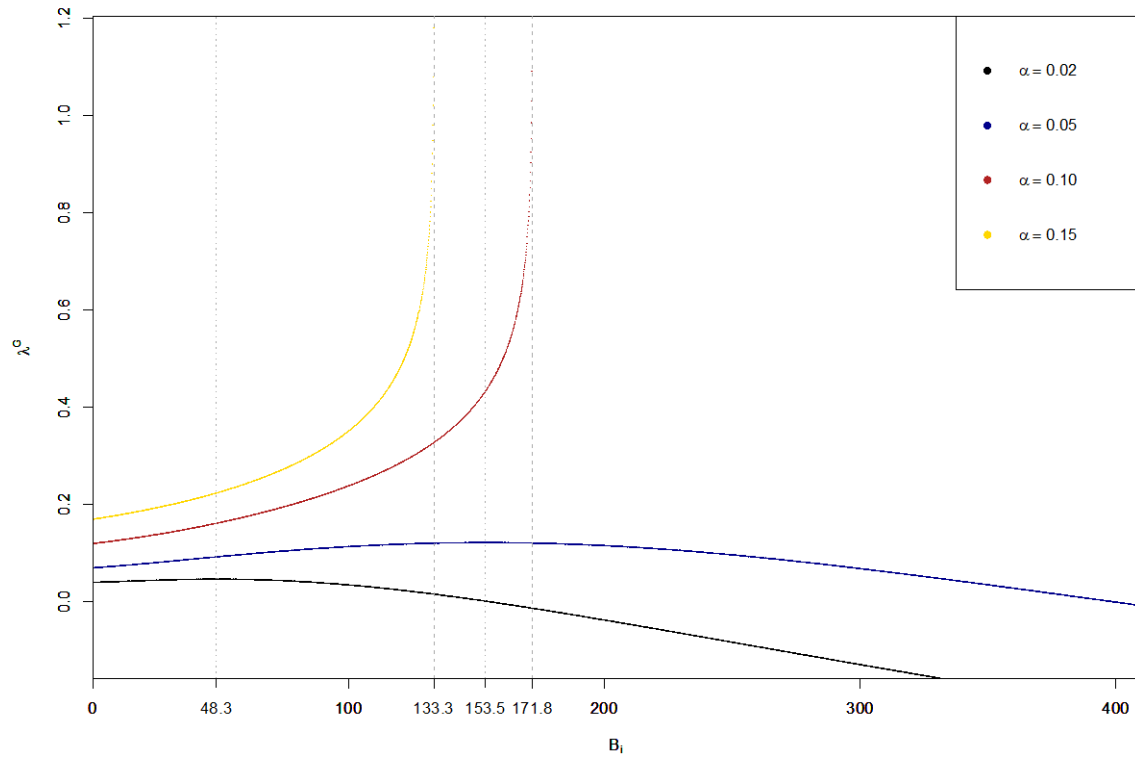**Changes in Leverage in Response to Changes in Key Parameter Values**

Figure 4

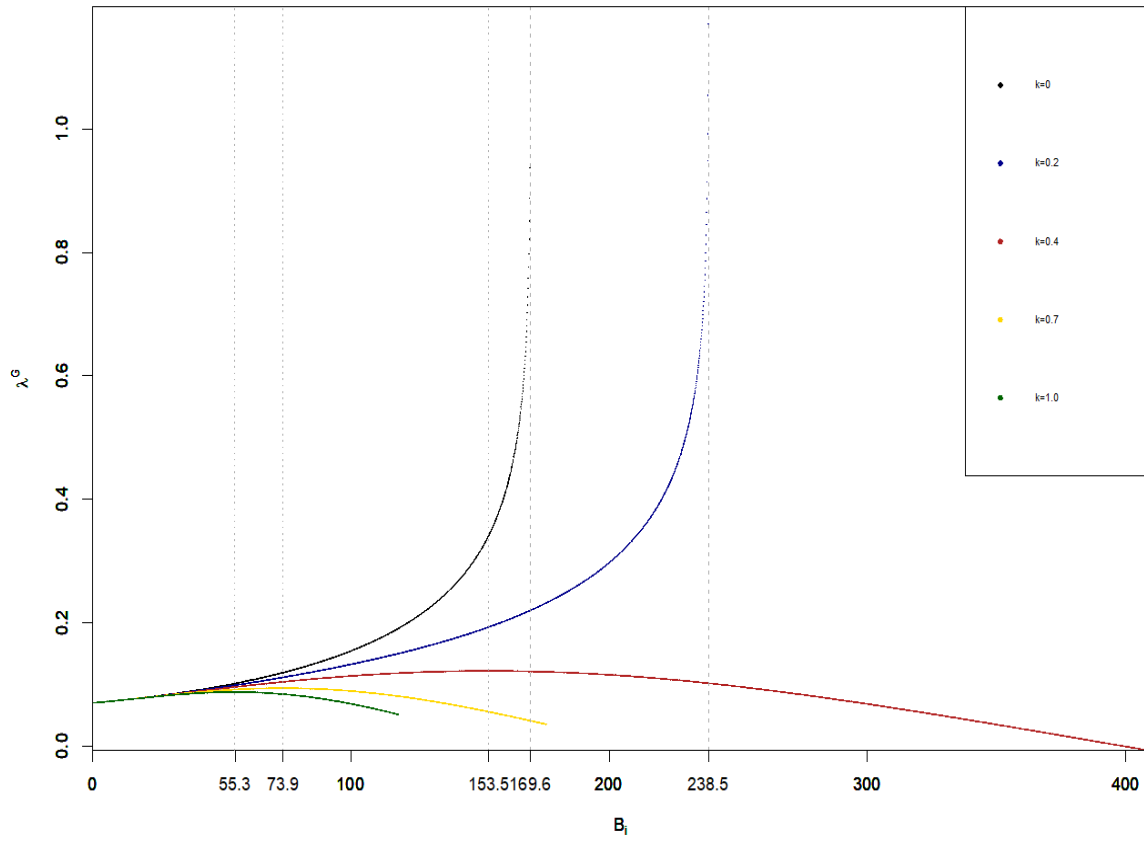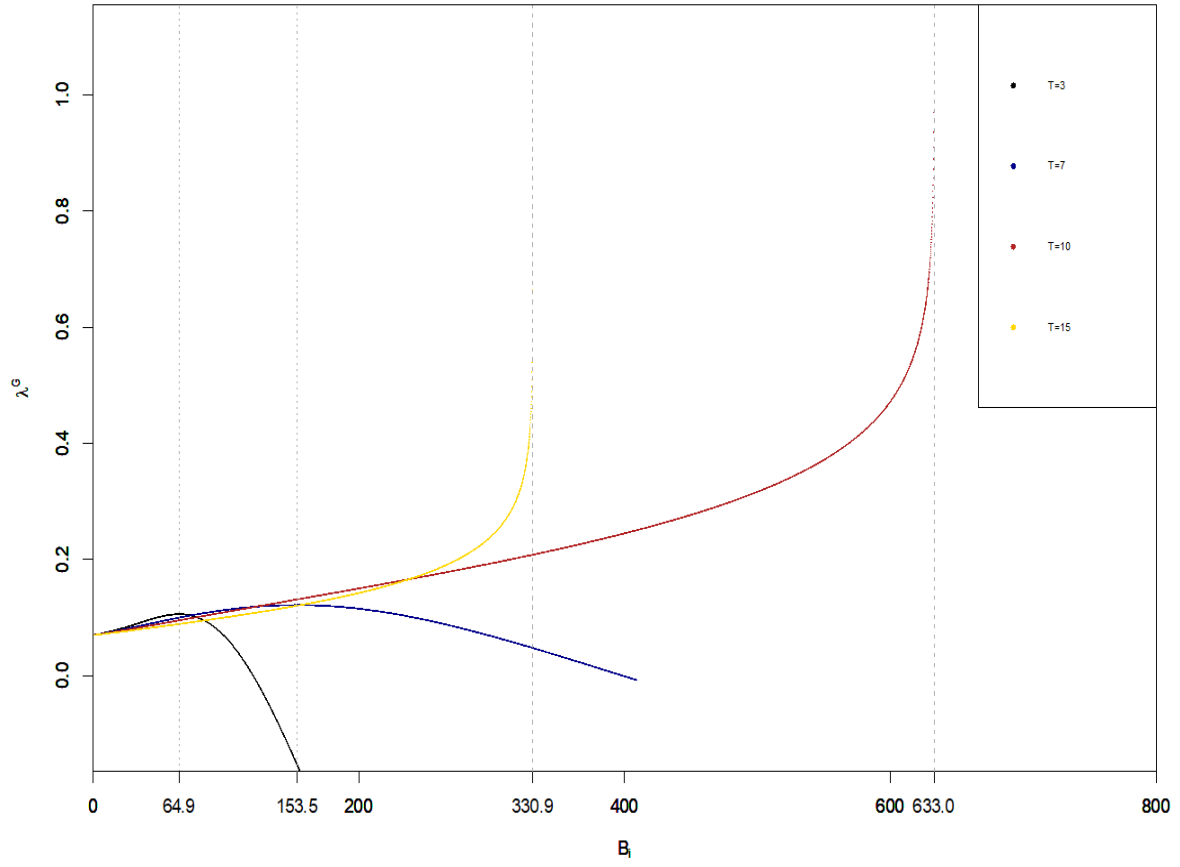## Leverage Choice and Target Rates of Return: Gross of Fees

Panel A - Base Case

Panel C – Alternative Costs of Financial Distress

Panel D – Alternative Fund-Debt Terms
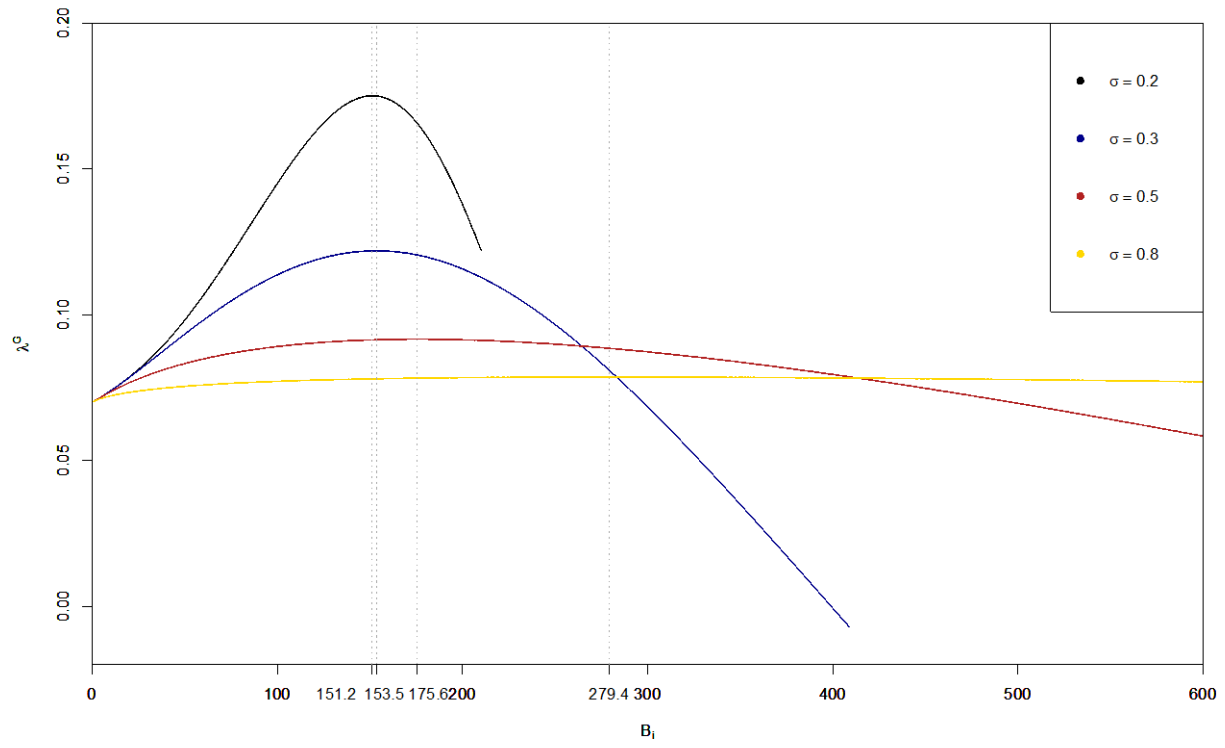
Panel E – Alternative Asset Volatilities

Figure 5

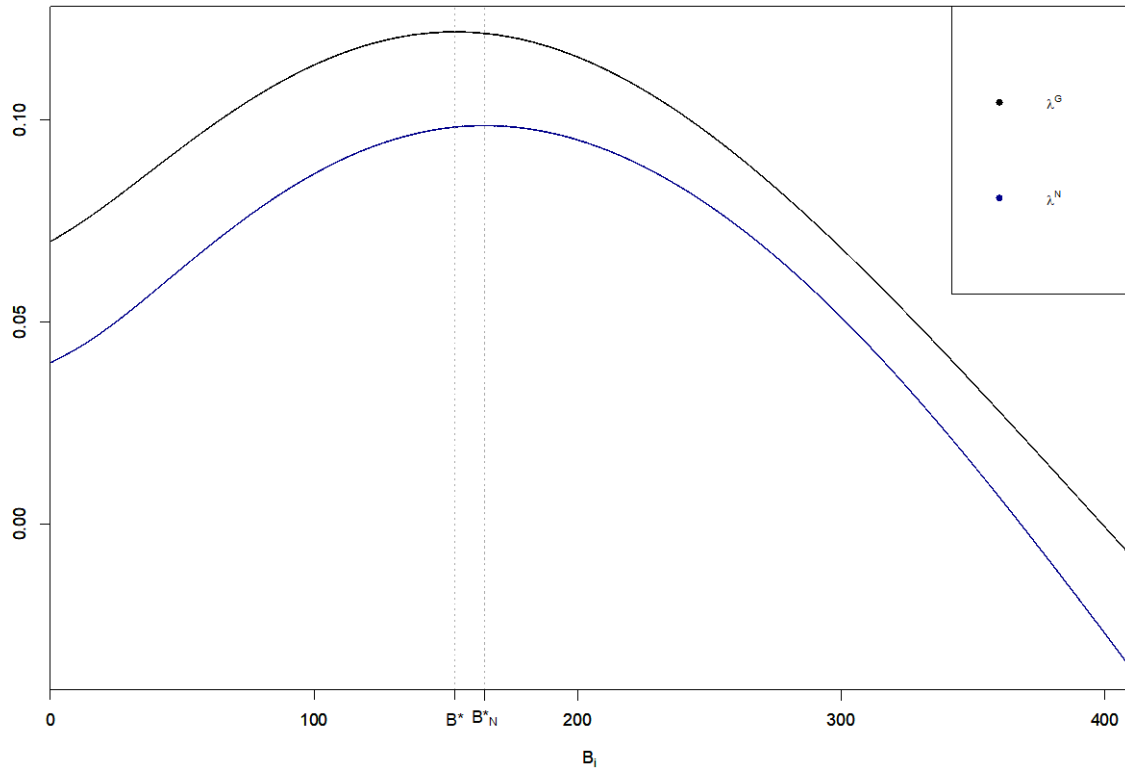**Leverage Choice and Target Rates of Return: Fee Regime #1**

Figure 6

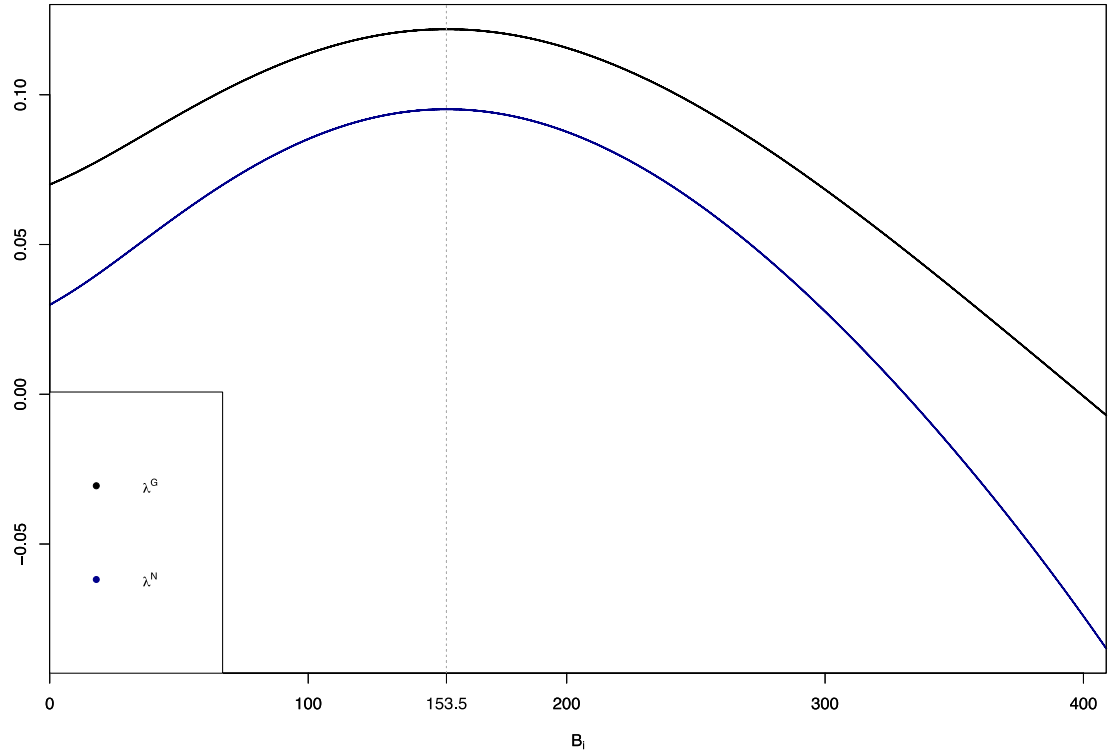**Leverage Choice and Target Rates of Return: Fee Regime #2**

Figure 7

Leverage Choice and Target Rates of Return: Fee Regime #3